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Weighted Sobolev Spaces on Curves

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In this paper we present a definition of weighted Sobolev spaces on curves and find general conditions under which the spaces are complete for non-closed compact curves. We also prove the density of the polynomials in these spaces and, finally, we find conditions under which the multiplication operator is bounded in the space of polynomials.

1. INTRODUCTION

In very different areas of mathematics ranging from the partial differential equations to approximation theory we find the topic of weighted Sobolev spaces (see, e.g. [HKM, K, Ku, KO, KS, T]). Some particular cases of Sobolev spaces with respect to measures instead of weights are studied in [EL, ELW1, ELW2], where we find some examples of Sobolev spaces related to ordinary differential equations and Sobolev orthogonal polynomials. We presented a very deep study of Sobolev spaces with respect to general measures in the real line in the papers [RARP1, RARP2, R1, R2, R3]. Now we are interested in Sobolev spaces with respect to general measures along curves in the complex plane.

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What we understand by a Sobolev norm on a Borel set $E \subseteq \mathbf{C}$ is the following: for $\mu = (\mu_0, \ldots, \mu_k)$ a vectorial Borel measure in E, the Sobolev norm in $W^{k,p}(E, \mu)$ of a function f which is holomorphic on a neighbourhood of E is defined by

$$\|f\|_{W^{k,p}(E,\mu)} := \left(\sum_{j=0}^{k} \|f^{(j)}\|^p_{L^p(E,\mu_j)} \right)^{1/p}.$$

Sobolev orthogonal polynomials on the unit circle and, more generally, on curves is a topic of recent and increasing interest in approximation theory; see, for example, [CM, FMP] (for the unit circle) and [BFM, M-F] (for the case of Jordan curves). If $E = \gamma$ is a simple and locally absolutely continuous curve, it is clear that the set of holomorphic functions whose norm in $W^{k,p}(\gamma, \mu)$ is finite is not a Banach space except when the support of μ is finite. In order to obtain a complete space we have to deal with functions which are not holomorphic. Consequently, we need to define $f^{(j)}$ when f is not holomorphic; the precise definition is presented in Section 2. In this context we talk about a Sobolev norm although it can be a seminorm; if this were the case we would take equivalence classes, as usual. When every polynomial has finite $W^{k,p}(\gamma, \mu)$-norm, we denote by $P^{k,p}(\gamma, \mu)$ the completion of polynomials with that norm.

The zeroes of the Sobolev orthogonal polynomials with respect to the scalar product in $W^{k,2}(\gamma, \mu)$ have been studied in [LP] in the case of a segment on the real line. There it is proved that they are contained in the disk $\{z \in \mathbf{C} : |z| \leqslant 2\|M\|\}$, where $(Mf)(x) = xf(x)$ is the multiplication operator, considered in the space $P^{k,2}([a,b], \mu)$. Consequently, the set of the zeroes of the Sobolev orthogonal polynomials is bounded if the multiplication operator is bounded. The location of these zeroes allows one to obtain results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [LP]). In [LP] they prove something more when they consider only sequentially dominated measures, i.e. measures such that $\#\operatorname{supp} \mu_0 = \infty$ and $d\mu_j = f_j \, d\mu_{j-1}$ with f_j bounded for $1 \leqslant j \leqslant k$. They prove that if μ is a finite sequentially dominated measure in $[a,b]$, then M is a bounded operator on $P^{k,2}([a,b], \mu)$. Recently, these results have been improved for measures on compact sets in $\mathbf{C}$ (see [LPP]).

It is not difficult to see that the multiplication operator can also be bounded when the vectorial measure is not sequentially dominated. In Section 8 other conditions are given in order to have the boundedness of M even on compact sets in $\mathbf{C}$. In [R1] one of the authors obtains a characterization of the boundedness of the operator M for measures in $\mathbf{R}$. Also, in Section 8 (see Theorem 8.1) this result is generalized for measures on compact sets in $\mathbf{C}$; therefore this theorem is useful in the study of orthogonal polynomials.

Though we do not have yet the definitions, we state the main theorems here. The results are numbered according to the section where they are proved. The first one gives a sufficient condition under which one obtains a complete Sobolev space. The condition is a bit technical although it is very general, so we prefer to state the theorem in a short version where this condition is denoted by: $(\gamma, \mu) \in \mathscr{C}$. The definition of the class $\mathscr{C}$ is in Section 4, Definition 4.2. The theorem is as follows:

THEOREM 5.1. *Let us consider* $1 \leqslant p \leqslant \infty$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a p-admissible vectorial measure in* γ *with* $(\gamma, \mu) \in \mathscr{C}$. *Then the Sobolev space* $W^{k,p}(\gamma, \mu)$ *is complete.*

Our main result on the density of polynomials in these spaces is Theorem 6.2. Now, the conditions we need are more restrictive than in Theorem 5.1, but we have found five general types of measures for which it is true. We simply name them by types 1, 2, 3, 4 and 5 and the definitions are in Section 6. These measures include the most usual examples like Jacobi-type weights (that are measures of type 2).

THEOREM 6.2. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a p-admissible vectorial measure in a non-closed compact curve* $\gamma : I \to \mathbf{C}$. *Assume that* $\gamma' \in W^{k-1,\infty}(I)$ *if* $k \geqslant 2$. *If* μ *is a measure of type* 1,2,3,4 *or* 5, *then P is dense in the Sobolev space* $W^{k,p}(\gamma, \mu)$.

The last result we present here is Theorem 8.1. It gives a necessary and sufficient condition so that the multiplication operator is bounded on the space $P^{k,p}(E, \mu)$. The kind of measures that appear here, extended sequentially dominated (ESD), is a generalization of sequentially dominated measures. The definition is in Section 5, Definition 5.1.

THEOREM 8.1. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a finite vectorial measure in a compact set E. Then, the multiplication operator is bounded in* $P^{k,p}(E, \mu)$ *if and only if there exists a vectorial measure* $\mu' \in \mathrm{ESD}$ *such that the Sobolev norms in* $W^{k,p}(E, \mu)$ *and* $W^{k,p}(E, \mu')$ *are comparable on P. Furthermore, we can choose* $\mu' = (\mu'_0, \dots, \mu'_k)$ *with* $\mu'_j := \mu_j + \mu_{j+1} + \cdots + \mu_k$.

We also answer (see Theorem 4.1) to the following main question: when the evaluation functional of f (or $f^{(j)}$) in a point is a bounded operator in $W^{k,p}(\gamma, \mu)$?

We also obtain results which partially generalize the classical result on density of polynomials in L^p of the unit circle to the context of Sobolev spaces (see Section 7).

Notation. We only consider simple curves which have a locally absolutely continuous parametrization. In the paper $k \geqslant 1$ denotes a fixed natural number; z_i are points along a curve $\gamma \subset \mathbf{C}$. All the measures we consider are Borel and positive, and all the weights are non-negative Borel measurable functions. We can split μ_j as $d\mu_j = d(\mu_j)_s + w_j\, ds$, where $(\mu_j)_s$ is singular with respect to the arc-length measure. w_j is a weight on γ and ds is the differential of arc-length. We always use this terminology for the Radon–Nikodym decomposition of μ_j. We identify a weight w on γ with the measure $w\, ds$. We denote by $\operatorname{supp} v$ the support of the measure v. If A is a Borel set in γ, $|A|$, $\chi_A, \bar{A}, \operatorname{int} A$ and $\#A$ denote, respectively, the length, the characteristic function, the closure, the interior and the cardinality of A (the interior and the closure of A are considered in the relative topology in γ). P and P_n denote, respectively, the set of all polynomials and the set of polynomials with degree less than or equal to n.

If $\gamma : I \to \mathbf{C}$ is a non-closed curve and $t_0 \in I$, by a right (respectively, left) neighbourhood of $z_0 = \gamma(t_0)$ in γ we mean the image by γ of $[t_0, t_0 + \varepsilon]$ (respectively, $[t_0 - \varepsilon, t_0]$) for some $\varepsilon > 0$. If t_0 is the maximum (respectively, minimum) of I we also have left (respectively, right) neighbourhoods of $\gamma(t_0)$.

If $\gamma : I \to \mathbf{C}$ is a closed curve and $t_0 \in I$, we can consider its periodic extension $\gamma_0 :\ \mathbf{R} \to \mathbf{C}$, and define left and right neighbourhoods of $\gamma(t_0)$ in a similar way.

Finally, the constants (denoted by c or c_i) in the formulae can change from line to line and even in the same line.

The outline of the paper is as follows. Sections 2–4 contain the definitions and some technical results we need. In Sections 5–7 we prove, respectively, the results on completeness, density and density in closed curves. We prove the results on the multiplication operator in Section 8.

2. DERIVATIVES ALONG CURVES

In this section we introduce a definition of *derivative along a curve* extending the usual complex derivative, which will be crucial in the future. As far as we know this concept is new. Recall that every curve is simple and has a locally absolutely continuous parametrization.

DEFINITION 2.1. (a) Let $I \subseteq \mathbf{R}$ be any interval and $\gamma : I \to \mathbf{C}$ be a curve. If z_1, z_2 are two distinct points of $\gamma(I)$, we denote by $\int_{z_1}^{z_2} g(\zeta)\, d\zeta$ the complex integral of the function g along the arc of γ joining z_1 and z_2, (which we denote by $[z_1, z_2]$). We also can consider arcs where one or the two extremal points are not included, as $(z_1, z_2), [z_1, z_2)$ *or* $(z_1, z_2]$. If γ is a closed curve we take the arc of γ joining z_1 and z_2 in the positive sense (according to the parametrization).

(b) Let z_0 be a fixed point in γ. If γ is compact we say that $f \in AC^k(\gamma)$ if f can be written as

$$f(z) = q(z) + \int_{z_0}^{z} h(\zeta)\,\frac{(z-\zeta)^{k-1}}{(k-1)!}d\zeta, \tag{2.1}$$

for some $h \in L^1(\gamma, ds)$ and some polynomial $q \in P_{k-1}$. If γ is a closed curve we require also the function $h \in L^1(\gamma, ds)$ to verify $\int_\gamma h(\zeta)\zeta^i\, d\zeta = 0$, for $0 \leqslant i < k$. When γ is not compact, we say that $f \in AC^k_{\text{loc}}(\gamma)$ if it can be split as in (2.1) with $h \in L^1_{\text{loc}}(\gamma, ds)$.

(c) If $f \in AC^k_{\text{loc}}(\gamma)$ and $z_0 \in \gamma$, we define its derivative f' along γ *as*

$$f'(z) = q'(z) + \int_{z_0}^{z} h(\zeta)\,\frac{(z-\zeta)^{k-2}}{(k-2)!}d\zeta,$$

where $q'(z)$ means the classical derivative of $q(z)$ and $\int_{z_0}^{z} h(\zeta)\,(z-\zeta)^{-1}/(-1)!\,d\zeta$ means $h(z)$.

Obviously, if γ is a compact real interval, the space $AC^1(\gamma)$ is the set of absolutely continuous functions in γ. If γ is a closed curve and $f \in AC^k(\gamma)$, we have $\int_\gamma h(\zeta)\,(z-\zeta)^{k-1}\,d\zeta = 0$ for every $z \in \gamma$. This property is equivalent to $f^{(j)}$ being continuous in γ for $0 \leqslant j < k$, where $f^{(j)}$ denotes the jth derivative (according to the previous definition) of f.

We also notice that it is natural to define the derivative along γ in this way, since this is the "inverse" of integration:

$$\begin{aligned}
\int_{z_0}^{z}\int_{z_0}^{\xi} h(\zeta)\,\frac{(\xi-\zeta)^{k-2}}{(k-2)!}\,d\zeta\,d\xi &= \int_{z_0}^{z}\int_{\zeta}^{z} h(\zeta)\,\frac{(\xi-\zeta)^{k-2}}{(k-2)!}\,d\xi\,d\zeta \\
&= \int_{z_0}^{z} h(\zeta)\left[\frac{(\xi-\zeta)^{k-1}}{(k-1)!}\right]_{\xi=\zeta}^{\xi=z} d\zeta \\
&= \int_{z_0}^{z} h(\zeta)\,\frac{(z-\zeta)^{k-1}}{(k-1)!}\,d\zeta.
\end{aligned}$$

Remark. Observe that if f is holomorphic in a region containing γ, then f' is the usual complex derivative of f at almost every point of γ.

Next, we prove that the definition of derivative is independent of the representation of f we are using. Without loss of generality we can assume that $\gamma' \neq 0$ almost everywhere since the definition of f' does not depend on the parametrization. In fact, we shall see that the representation is unique. Let us suppose that

$$f(z) = q(z) + H_k(z) = r(z) + G_k(z),$$

where $q(z)$ and $r(z)$ are polynomials with degree at most $k-1$ and

$$H_k(z) = \int_{z_0}^{z} h(\zeta)\,\frac{(z-\zeta)^{k-1}}{(k-1)!}\,d\zeta, \qquad G_k(z) = \int_{z_0}^{z} g(\zeta)\,\frac{(z-\zeta)^{k-1}}{(k-1)!}\,d\zeta.$$

We want to see that $q = r$ and $g = h$. Observe that

$$\int_{z_0}^{z} (g-h)\,(\zeta)\,\frac{(z-\zeta)^{k-1}}{(k-1)!}\,d\zeta = (q-r)\,(z).$$

We proceed now by induction. Let us denote $v = g - h$. For $k = 1$, the function $V(z) = \int_{z_0}^{z} v(\zeta)\,d\zeta$ is constant. It follows that $\int_{t_1}^{t_2} v(\gamma(t))\gamma'(t)\,dt = 0$ for all $t_1, t_2 \in I$ and this implies that $v(\gamma(t)) = 0$ almost everywhere in I. Therefore $v = 0$, i.e., $g = h$ and $q = r$.

Suppose now that

$$\int_{z_0}^{z} v(\zeta)\,\frac{(z-\zeta)^{n}}{n!}\,d\zeta$$

is a polynomial of degree at most n if and only if $v = 0$, and consider the function $V \in P_{n+1}$ defined by

$$V(z) = \int_{z_0}^{z} v(\zeta)\,\frac{(z-\zeta)^{n+1}}{(n+1)!}\,d\zeta.$$

If $z_0 = \gamma(t_0),\ z = \gamma(T)$, then

$$W(T) := V(\gamma(T)) = \int_{t_0}^{T} v(\gamma(t))\,\frac{(\gamma(T)-\gamma(t))^{n+1}}{(n+1)!}\gamma'(t)\,dt,$$

and therefore

$$V'(z)\gamma'(T) = W'(T) = \int_{t_0}^{T} v(\gamma(t))\,\frac{(\gamma(T)-\gamma(t))^{n}}{n!}\,\gamma'(t)\gamma'(T)\,dt.$$

As $\gamma' \neq 0$ almost everywhere it follows that

$$V'(z) = \int_{z_0}^{z} v(\zeta) \frac{(z-\zeta)^n}{n!} \, d\zeta$$

almost everywhere and so everywhere by continuity. Since $V' \in P_n$, the induction hypothesis implies that $v = 0$.

We need to prove now that this definition does not depend on the choice of the point z_0. To see this, let us denote

$$H_{k,z_0}(z) = \int_{z_0}^{z} h(\zeta) \frac{(z-\zeta)^{k-1}}{(k-1)!} \, d\zeta.$$

If z_1 is another point in γ, then

$$H_{k,z_0}(z) = \int_{z_0}^{z_1} h(\zeta) \frac{(z-\zeta)^{k-1}}{(k-1)!} \, d\zeta + \int_{z_1}^{z} h(\zeta) \frac{(z-\zeta)^{k-1}}{(k-1)!} d\zeta = Q_k(z) + H_{k,z_1}(z), \tag{2.2}$$

where $Q_k \in P_{k-1}$. Observe that (2.2) is true for a closed curve γ since then $\int_\gamma h(\zeta)\,(z-\zeta)^{k-1}\,d\zeta = 0$ for every $z \in \gamma$. Consequently, $H'_{k,z_0} = Q'_k + H'_{z,k_1}$. Therefore, in what follows, we can assume that z_0 is arbitrary but fixed.

Finally, we need also to prove that our definition does not depend on k. Indeed, we shall show that if $f \in AC^k_{\mathrm{loc}}(\gamma)$ then $f \in AC^{k-1}_{\mathrm{loc}}(\gamma)$ and the corresponding definitions of derivative along γ coincide. Let us suppose that

$$f(z) = q(z) + \int_{z_0}^{z} h(\zeta) \frac{(z-\zeta)^{k-1}}{(k-1)!} \, d\zeta \quad \text{and}$$
$$f(z) = Q(z) + \int_{z_0}^{z} H(\zeta) \frac{(z-\zeta)^{k-2}}{(k-2)!} \, d\zeta,$$

with $q \in P_{k-1}$ and $Q \in P_{k-2}$. Then we can write $q(z) = q_0(z) + \ell(z-z_0)^{k-1}/(k-1)!$ with $q_0 \in P_{k-2}$ and therefore, integrating by parts in the first integral,

$$u = \frac{(z-\zeta)^{k-1}}{(k-1)!}, \qquad du = -\frac{(z-\zeta)^{k-2}}{(k-2)!} \, d\zeta,$$

$$dv = h(\zeta)\, d\zeta, \qquad v(\zeta) = \int_{z_0}^{\zeta} h(\xi)\, d\xi + \ell,$$

and so

$$f(z) = q_0(z) + \ell\,\frac{(z-z_0)^{k-1}}{(k-1)!} + \left[\frac{(z-\zeta)^{k-1}}{(k-1)!}\,v(\zeta)\right]_{\zeta=z_0}^{\zeta=z} + \int_{z_0}^{z} v(\zeta)\frac{(z-\zeta)^{k-2}}{(k-2)!}\,d\zeta$$

$$= q_0(z) + \int_{z_0}^{z} v(\zeta)\,\frac{(z-\zeta)^{k-2}}{(k-2)!}\,d\zeta.$$

This means, by the unicity of the representation for the same k, that $q_0 = Q$ and $v = H$. On the other hand, integrating by parts again, we have that

$$\begin{aligned} f'(z) &= \left(q(z) + \int_{z_0}^{z} h(\zeta)\frac{(z-\zeta)^{k-1}}{(k-1)!}\,d\zeta\right)' \\ &= q_0'(z) + \ell\,\frac{(z-z_0)^{k-2}}{(k-2)!} + \int_{z_0}^{z} h(\zeta)\,\frac{(z-\zeta)^{k-2}}{(k-2)!}\,d\zeta \\ &= q_0'(z) + \int_{z_0}^{z} v(\zeta)\,\frac{(z-\zeta)^{k-3}}{(k-3)!}\,d\zeta \\ &= \left(Q(z) + \int_{z_0}^{z} H(\zeta)\,\frac{(z-\zeta)^{k-2}}{(k-2)!}\,d\zeta\right)'. \end{aligned}$$

The proof of the following three results is trivial.

LEMMA 2.1. *If $f, g \in AC_{\mathrm{loc}}^{k}(\gamma)$ and $\alpha, \beta \in \mathbf{C}$, then $\alpha f + \beta g \in AC_{\mathrm{loc}}^{k}(\gamma)$.*

LEMMA 2.2. *$f \in AC_{\mathrm{loc}}^{k}(\gamma)$ if and only if the jth derivative $f^{(j)}$ along γ belongs to $AC_{\mathrm{loc}}^{k-j}(\gamma)$.*

LEMMA 2.3. *If $f \in AC_{\mathrm{loc}}^{k}(\gamma)$ and $z_0 \in \gamma$, then*

$$f(z) = q(z) + \int_{z_0}^{z} h(\zeta)\,\frac{(z-\zeta)^{k-1}}{(k-1)!}\,d\zeta,$$

where $q(z)$ is the $(k-1)$th Taylor polynomial of f centered at z_0, i.e.,

$$q(z) = \sum_{j=0}^{k-1} \frac{f^{(j)}(z_0)}{j!}\,(z-z_0)^j \quad \text{and} \quad h(z) = f^{(k)}(z).$$

DEFINITION 2.2. We say that $f \in C^k(\gamma)$ if $f \in AC_{\mathrm{loc}}^{k}(\gamma)$ and $f^{(k)}$ is continuous in γ.

Next, we study Leibniz' rule.

LEMMA 2.4. *If* $F, G \in AC^1_{\mathrm{loc}}(\gamma)$ *then* $FG \in AC^1_{\mathrm{loc}}(\gamma)$ *and* $(FG)' = F'G + FG'$.

Proof. We can write

$$F(z) = F(z_0) + \int_{z_0}^{z} f(\zeta)\,d\zeta, \qquad G(z) = G(z_0) + \int_{z_0}^{z} g(\zeta)\,d\zeta,$$

where $f, g \in L^1_{\mathrm{loc}}(\gamma, ds)$ and $z, z_0 \in \gamma$, but,

$$\begin{aligned} F(z)\,G(z) = {} & F(z_0)\,G(z_0) + \int_{z_0}^{z} (F(z_0)g(\zeta) + G(z_0)f(\zeta))\,d\zeta \\ & + \left(\int_{z_0}^{z} f(\zeta)\,d\zeta\right)\left(\int_{z_0}^{z} g(\zeta)\,d\zeta\right) \end{aligned} \tag{2.3}$$

and applying Fubini's Theorem we get

$$\begin{aligned} \left(\int_{z_0}^{z} f(\xi)\,d\xi\right)\left(\int_{z_0}^{z} g(\zeta)\,d\zeta\right) &= \int_{z_0}^{z}\int_{z_0}^{z} f(\xi)\,g(\zeta)\,d\xi\,d\zeta \\ &= \int_{z_0}^{z}\int_{z_0}^{\xi} f(\xi)\,g(\zeta)\,d\zeta\,d\xi \\ &\quad + \int_{z_0}^{z}\int_{\xi}^{z} f(\xi)\,g(\zeta)\,d\zeta\,d\xi \\ &= \int_{z_0}^{z}\int_{z_0}^{\xi} f(\xi)\,g(\zeta)\,d\zeta\,d\xi \\ &\quad + \int_{z_0}^{z}\int_{z_0}^{\zeta} f(\xi)\,g(\zeta)\,d\xi\,d\zeta \\ &= \int_{z_0}^{z}\int_{z_0}^{\xi} f(\xi)\,g(\zeta)\,d\zeta\,d\xi \\ &\quad + \int_{z_0}^{z}\int_{z_0}^{\zeta} f(\xi)\,g(\zeta)\,d\xi\,d\zeta \\ &= \int_{z_0}^{z} f(\xi)\,(G(\xi) - G(z_0))\,d\xi \\ &\quad + \int_{z_0}^{z} g(\zeta)\,(F(\zeta) - F(z_0))\,d\zeta. \end{aligned}$$

This and (2.3) give

$$F(z)\,G(z) = F(z_0)\,G(z_0) + \int_{z_0}^{z} (F(\zeta)\,g(\zeta) + G(\zeta)f(\zeta))\,d\zeta,$$

with $Fg + Gf \in L^1_{\text{loc}}(\gamma, ds)$, i.e., $FG \in AC^1_{\text{loc}}(\gamma)$ and $(FG)' = F'G + FG'$ almost everywhere in γ. ∎

Proceeding inductively we obtain that if $F, G \in AC^k_{\text{loc}}(\gamma)$ then $(FG)^{(k-1)} \in AC^k_{\text{loc}}(\gamma)$ which implies that $FG \in AC^k_{\text{loc}}(\gamma)$, that is

LEMMA 2.5. *Let* $F, G \in AC^k_{\text{loc}}(\gamma)$. *Then* $FG \in AC^k_{\text{loc}}(\gamma)$ *and verifies Leibniz' rule, i.e.,*

$$(FG)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} F^{(j)} G^{(k-j)}.$$

LEMMA 2.6. *Let us consider* $\gamma : I \to \mathbf{C}$ *with* $\gamma' \neq 0$ *almost everywhere. Then* $f \in AC^1_{\text{loc}}(\gamma)$ *if and only if* $f \circ \gamma \in AC^1_{\text{loc}}(I)$. *Furthermore, if* $f \in AC^1_{\text{loc}}(\gamma)$ *we have*

$$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\gamma'(t) \qquad \textit{for almost every } t \in I.$$

Proof. If $f \in AC^1_{\text{loc}}(\gamma)$ we obtain directly $f \circ \gamma \in AC_{\text{loc}}(I)$. Fix now $t_0 \in I$. If $f \circ \gamma \in AC_{\text{loc}}(I)$ then $d(f \circ \gamma)/dt \in L^1_{\text{loc}}(I)$ and so

$$f(\gamma(t)) = f(\gamma(t_0)) + \int_{t_0}^{t} \frac{1}{\gamma'(\tau)} \frac{d}{d\tau} (f(\gamma(\tau)))\gamma'(\tau)\,d\tau,$$

for every $t \in I$. Therefore, for every $z \in \gamma$,

$$f(z) = f(z_0) + \int_{z_0}^{z} h(\zeta)\,d\zeta, \qquad \text{with } h(\zeta) = \left(\frac{1}{\gamma'(\tau)} \frac{d}{d\tau}(f(\gamma(\tau)))\right)(\gamma^{-1}(\zeta)).$$ ∎

Finally, let us introduce our last concept on derivatives:

DEFINITION 2.3. We define the D-derivative of a function g in I, as

$$D[g](t) = \frac{g'(t)}{\gamma'(t)} \qquad \text{and} \qquad D^k = D^{k-1} \circ D.$$

It is natural to ask what functions belong to the class $AC^k_{\text{loc}}(\gamma)$. The following results answer this question if γ is smooth enough.

LEMMA 2.7. *Let us suppose that* $\gamma \in AC^k_{\mathrm{loc}}(I)$ *and* $\gamma' \neq 0$ *in* I. *Then* $f \in AC^k_{\mathrm{loc}}(\gamma)$ *if and only if* $f \circ \gamma \in AC^k_{\mathrm{loc}}(I)$. *Furthermore, if* $f \in AC^k_{\mathrm{loc}}(\gamma)$ *we have*

$$D^j[f \circ \gamma](t) = f^{(j)}(\gamma(t)) \qquad \textit{for} \quad 1 \leqslant j \leqslant k \textit{ and almost every } t \in I. \tag{2.4}$$

Proof. Assume that $f \circ \gamma \in AC^k_{\mathrm{loc}}(I)$ and fix $t_0 \in I$. Lemma 2.6 gives

$$f(\gamma(t)) = f(\gamma(t_0)) + \int_{t_0}^{t} D[f \circ \gamma](\tau)\gamma'(\tau)\, d\tau.$$

Integrating by parts, we have, for $1 \leqslant j < k$, that

$$\begin{aligned}\int_{t_0}^{t} & D^j[f \circ \gamma](\tau)\, \frac{(\gamma(t) - \gamma(\tau))^{j-1}}{(j-1)!}\, \gamma'(\tau)\, d\tau \\ &= D^j[f \circ \gamma](t_0) \frac{(\gamma(t) - \gamma(t_0))^j}{j!} + \int_{t_0}^{t} D^{j+1}[f \circ \gamma](\tau)\, \frac{(\gamma(t) - \gamma(\tau))^j}{j!}\, \gamma'(\tau)\, d\tau.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned} f(\gamma(t)) = & \sum_{j=0}^{k-1} D^j[f \circ \gamma](t_0)\, \frac{(\gamma(t) - \gamma(t_0))^j}{j!} \\ & + \int_{t_0}^{t} D^k[f \circ \gamma](\tau)\, \frac{(\gamma(t) - \gamma(\tau))^{k-1}}{(k-1)!}\, \gamma'(\tau)\, d\tau.\end{aligned}$$

Then $f \in AC^k_{\mathrm{loc}}(\gamma)$ and we have (2.4). ∎

Assume now that $f \in AC^k_{\mathrm{loc}}(\gamma)$. We prove (2.4) by induction in j. Lemma 2.6 gives $D[f \circ \gamma](t) = f'(\gamma(t))$. Assume that $D^j[f \circ \gamma](t) = f^{(j)}(\gamma(t))$ for some j $(1 \leqslant j < k)$. Since $f^{(j)} \in AC^1_{\mathrm{loc}}(\gamma)$ we have by Lemma 2.6 that $f^{(j)}(\gamma(t)) = D^j[f \circ \gamma](t) \in AC^1_{\mathrm{loc}}(I)$ and

$$\frac{d}{dt}(D^j[f \circ \gamma](t)) = f^{(j+1)}(\gamma(t))\gamma'(t).$$

Therefore $D^{j+1}[f \circ \gamma](t) = f^{(j+1)}(\gamma(t))$. This gives (2.4). Now it is immediate that

$$D^{k-1}[f \circ \gamma](t) = \frac{(f \circ \gamma)^{(k-1)}(t)}{\gamma'(t)^{k-1}} + \frac{Q[(f \circ \gamma)', \dots, (f \circ \gamma)^{(k-2)}, \gamma', \dots, \gamma^{(k-1)}](t)}{\gamma'(t)^{2k-1}}, \tag{2.5}$$

where Q is a polynomial. Since $f^{(k-1)} \in AC^1_{\mathrm{loc}}(\gamma)$, Lemma 2.6 gives $D^{k-1}[f \circ \gamma](t) = f^{(k-1)}(\gamma(t)) \in AC^1_{\mathrm{loc}}(I)$. This fact and (2.5) give $(f \circ \gamma)^{(k-1)} \in AC^1_{\mathrm{loc}}(I)$.

Corollary 2.1. *Assume that $\gamma \in C^k(I)$ and $\gamma' \neq 0$ in I. Then $f \in C^k(\gamma)$ if and only if $f \circ \gamma \in C^k(I)$.*

3. SOBOLEV SPACES

Obviously one of our main problems is to define the space $W^{k,p}(\gamma.\mu)$. There are two natural definitions:

(1) $W^{k,p}(\gamma, \mu)$ is the biggest space of (classes of) functions f regular enough with $\|f\|_{W^{k,p}(\gamma,\mu)} < \infty$.

(2) $W^{k,p}(\gamma, \mu)$ is the closure of a good set of functions (e.g. $C^\infty(\gamma)$ or P) with the norm $\|\cdot\|_{W^{k,p}(\gamma,\mu)}$.

However both approaches have serious difficulties:

We consider first approach (1). It is clear that the derivatives $f^{(j)}$ must be derivatives along γ in order to obtain a complete Sobolev space. Therefore, we need to restrict the measures μ to a class of p-admissible measures (see Definition 3.6). Roughly speaking, μ is p-admissible if $(\mu_j)_s$, for $1 \leqslant j \leqslant k$, is concentrated in the set of points where $f^{(j)}$ is continuous, for every function f of the space; otherwise $f^{(j)}$ is determined, up to zero-Lebesgue measure sets. Then $(\mu_k)_s$ is identically zero. However, there is no restriction on the support of $(\mu_0)_s$.

This reasonable approach excludes norms appearing in the theory of Sobolev orthogonal polynomials. Even if we work with the simpler case of the weighted Sobolev spaces $W^{k,p}(\gamma, w)$ (measures without singular part) we must impose the condition that w_j belongs to the class B_p (see Definition 3.2) in order to have a complete weighted Sobolev space (see, e.g. [KO]).

Approach (2) is simpler: we know that the completion of every normed space exists (e.g. $(C^\infty(\gamma), \|\cdot\|_{W^{k,p}(\gamma,\mu)})$ or $(P, \|\cdot\|_{W^{k,p}(\gamma,\mu)})$), but we have two difficulties. The first one is evident: we do not get an explicit description of the Sobolev functions as in (1) (in Section 6 there are several theorems which prove that both definitions of Sobolev space are the same for p-admissible measures). The second problem is worse: The completion of a normed space is by definition a set of equivalence classes of Cauchy sequences. In many cases this completion is not a function space (see [R1, Theorem 3.1 and its Remark]).

However, since we need to work with the multiplication operator in $P^{k,p}(\gamma, \mu)$, we have to choose this second approach if μ is not p-admissible. First of all, we explain the definition of generalized Sobolev space on a curve. Let us start with some preliminary technical definitions.

DEFINITION 3.1. We say that two functions u, v are comparable on the set $A \subseteq \gamma$ if there are positive constants c_1, c_2 such that $c_1 v(x) \leqslant u(x) \leqslant c_2 v(x)$ for almost every $x \in A$. Since measures and norms are functions on measurable sets and vectors, respectively, we can talk about comparable measures and comparable norms. We say that two vectorial weights or vectorial measures are comparable if each component is comparable.

In what follow, the symbol $a \asymp b$ means that a and b are comparable for a and b functions, measures or norms.

Obviously, the spaces $L^p(A, \mu)$ and $L^p(A, v)$ are the same and have comparable norms if μ and v are comparable on A. Therefore, in order to obtain our results we can replace a measure μ by any comparable measure v.

To define a Sobolev space along a curve γ we consider first a class of weights which plays an important role in our results.

DEFINITION 3.2. If $1 \leqslant p \leqslant \infty$, we say that a weight w belongs to $B_p([z_1, z_2])$ if and only if

$$w^{-1} \in L^{1/(p-1)}([z_1, z_2]) \qquad \text{if } p < \infty,$$

$$w^{-1} \in L^{1}([z_1, z_2]) \qquad \text{if } p = \infty.$$

Also, if J is any arc of γ we say that $w \in B_p(J)$ if $w \in B_p(J_0)$ for every compact arc $J_0 \subseteq J$. We say that a weight belongs to $B_p(J)$, where J is a union of disjoint arcs $\bigcup_{i \in A} J_i$, if it belongs to $B_p(J_i)$, for $i \in A$.

If the curve γ is $\mathbf{R}$, then $B_p(\mathbf{R})$ contains the classical $A_p(\mathbf{R})$ weights appearing in Harmonic Analysis (see [Mu1] or [GR]). The classes $B_p(\Omega)$, with $\Omega \subseteq \mathbf{R}^n$ and $A_p(\mathbf{R}^n)(1 < p < \infty)$ have been used in other definitions of weighted Sobolev spaces on $\mathbf{R}^n$ in [KO, K], respectively.

DEFINITION 3.3. Let us consider $1 \leqslant p \leqslant \infty$ and a vectorial measure $\mu = (\mu_0, \ldots, \mu_k)$ defined on the curve γ. For $0 \leqslant j \leqslant k$ we define the open set

$$\Omega_j := \{z \in \gamma : \exists \text{ an open neighbourhood } V \text{ of } z \text{ on the curve } \gamma \text{ with } w_j \in B_p(V)\}.$$

Remark. Observe that we always have $w_j \in B_p(\Omega_j)$ for any $1 \leqslant p \leqslant \infty$ and $0 \leqslant j \leqslant k$. In fact, Ω_j is the greatest open set U with $w_j \in B_p(U)$. Obviously,

Ω_j depends on μ and p, although p and μ do not appear explicitly in the symbol Ω_j. Applying Hölder inequality it is easy to check that if $f^{(j)} \in L^p(\Omega_j, w_j)$ with $1 \leqslant j \leqslant k$, then $f^{(j)} \in L^1_{\text{loc}}(\Omega_j)$ and $f^{(j-1)} \in AC^1_{\text{loc}}(\Omega_j)$.

The following definitions also depend on μ and p, although μ and p do not appear explicitly.

Let us consider $1 \leqslant p \leqslant \infty$, a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ and $z_0 \in \gamma$. We can modify the measure μ in a neighbourhood of z_0, using the following version of Muckenhoupt inequality in curves. This modified measure is equivalent in some sense to the original one (see Theorem 4.1).

THEOREM 3.1 (Muckenhoupt Inequality in Curves). *Let us consider* $1 \leqslant p \leqslant \infty$, $[z_0, z_1] \subseteq \gamma$ *and* μ_0, μ_1 *measures in* $(z_0, z_1]$. *Assume also* $(\mu_0)_s \equiv 0$ *if* $p = \infty$. *Then there exists a positive constant c such that*

$$\left\| \int_z^{z_1} g(\zeta)\, d\zeta \right\|_{L^p((z_0,z_1],\mu_0)} \leqslant c \|g\|_{L^p((z_0,z_1],\mu_1)} \tag{3.1}$$

for any measurable function g in $(z_0, z_1]$, *if and only if*

$$\sup_{\zeta \in (z_0,z_1)} \mu_0((z_0,\zeta])\|w_1^{-1}\|_{L^{1/(p-1)}([\zeta,z_1])} < \infty \qquad \text{if } 1 \leqslant p < \infty,$$
$$\operatorname*{ess\,sup}_{\zeta \in (z_0,z_1)} w_0(\zeta) \int_\zeta^{z_1} w_1(\xi)^{-1} |d\xi| < \infty \qquad \text{if } p = \infty, \tag{3.2}$$

where ess sup *refers to the arc-length.*

Remark. This inequality is already known if γ is contained in the real line (see [Mu2, M, p. 44] for $1 \leqslant p < \infty$, and [RARP1, Lemma 3.2] for $p = \infty$).

Proof. We only deal with the case $1 < p < \infty$; the cases $p = 1$ and $p = \infty$ are similar. Consider the arc-length parametrization $\gamma : [a, b] \to [z_0, z_1]$.

We prove first that (3.2) implies (3.1). We can define measures μ_0^*, μ_1^* $(a, b]$ as follows: $\mu_i^*(D) = \mu_i(\gamma(D))$ for any Borel subset D of $(a, b]$ and for $i = 0, 1$. Consequently, $\int_{z_1}^{z_2} f\, d\mu_i = \int_a^b (f \circ \gamma)\, d\mu_i^*$ for any $f \in L^1(\gamma, \mu_i)$. Note that w_1^*, the absolutely continuous part of μ_1^*, is equal to $w_1 \circ \gamma$ almost everywhere, since $|\gamma'| = 1$ almost everywhere. By condition (3.2) we have

$$\sup_{t \in (a,b)} \mu_0^*((a,t])\|(w_1^*)^{-1}\|_{L^{1/(p-1)}(t,b])} < \infty,$$

since $w_1^* = w_1 \circ \gamma$ and $|\gamma'| = 1$ almost everywhere. Muckenhoupt inequality in the real line gives

$$\left|\left|\int_t^b |g(\gamma(\tau))| \, d\tau\right|\right|_{L^p((a,b],\mu_0^*)} \leqslant c||g \circ \gamma||_{L^p((a,b],\mu_1^*)},$$

for any measurable function g defined in $(z_0, z_1]$. This inequality and the facts $||g \circ \gamma||_{L^p((a,b],\mu_1^*)} = ||g||_{L^p((z_0,z_1],\mu_1)}$ and

$$\begin{aligned}\left|\left|\int_z^{z_1} g(\zeta) \, d\zeta\right|\right|_{L^p((z_0,z_1],\mu_0)} &= \left|\left|\int_{\gamma(t)}^{\gamma(b)} g(\zeta) \, d\zeta\right|\right|_{L^p((a,b],\mu_0^*)} \\ &= \left|\left|\int_t^b g(\gamma(\tau))\gamma'(\tau) \, d\tau\right|\right|_{L^p((a,b],\mu_0^*)} \\ &\leqslant \left|\left|\int_t^b |g(\gamma(\tau))| \, d\tau\right|\right|_{L^p((a,b],\mu_0^*)},\end{aligned}$$

give (3.1).

Assume now (3.1). Fix $\zeta \in (z_0, z_1)$ and consider the function g in $(z_0, z_1]$ defined by

$$g(z) := w_1(z)^{-1/(p-1)} \chi_{[\zeta,z_1]\setminus A}(z) \overline{\gamma'(\gamma^{-1}(z))},$$

if $w_1 \in B_p((z_0, z_1])$, where A is a set of zero length in $(z_0, z_1]$ with $(\mu_1)_s$ concentrated in A. If $w_1 \notin B_p((z_0, z_1])$, we can consider $w_1 + \varepsilon$ instead of w_1 and take the limit as $\varepsilon \to 0^+$. We have

$$||g||^p_{L^p((z_0,z_1],\mu_1)} = \int_\zeta^{z_1} w_1(z)^{-1/(p-1)} |dz| = ||w_1^{-1}||^{1/(p-1)]}_{L^{1/(p-1)}([\zeta,z_1])} \tag{3.3}$$

and

$$\begin{aligned}\left|\left|\int_z^{z_1} g(\xi) \, d\xi\right|\right|^p_{L^p((z_0,z_1],\mu_0)} &\geqslant \int_{z_0}^{\zeta} \left|\int_z^{z_1} g(\xi) \, d\xi\right|^p d\mu_0(z) \\ &= \int_{z_0}^{\zeta} \left|\int_\zeta^{z_1} g(\xi) \, d\xi\right|^p d\mu_0(z) \\ &= \mu_0((z_0, \zeta]) ||w_1^{-1}||^{p/(p-1)}_{L^{1/(p-1)}(\zeta,z_1])},\end{aligned} \tag{3.4}$$

since

$$\int_{\zeta}^{z_1} g(\xi)\,d\xi = \int_t^b g(\gamma(\tau))\gamma'(\tau)\,d\tau$$
$$= \int_t^b w_1(\gamma(\tau))^{-1/(p-1)}\overline{\gamma'(\tau)}\,\gamma'(\tau)\,d\tau = \int_{\zeta}^{z_1} w_1(\xi)^{-1/(p-1)}|d\xi|,$$

if $\gamma(t) = \zeta$. Now (3.1), (3.3) and (3.4) give (3.2). ∎

Definition 3.4. A vectorial measure $\overline{\mu} = (\overline{\mu}_0, \ldots, \overline{\mu}_k)$ is a right completion of a vectorial measure $\mu = (\mu_0, \ldots, \mu_k)$ with respect to $z_0 \in \gamma$ in a right neighbourhood $[z_0, z_1]$, if $\overline{\mu}_k = \mu_k$ in γ, $\overline{\mu}_j = \mu_j$ in the complement of $(z_0, z_1]$ and

$$\overline{\mu}_j = \mu_j + \tilde{\mu}_j \qquad \text{in } (z_0, z_1] \text{ for } 0 \leqslant j < k,$$

where $\tilde{\mu}_j$ is any measure satisfying:

(i) $\tilde{\mu}_j((z_0, z_1]) < \infty$ if $1 \leqslant p < \infty$,

(ii) $(\tilde{\mu}_j)_s \equiv 0$ and $\tilde{w}_j \in L^\infty([z_0, z_1])$ if $p = \infty$,

(iii) $\Lambda_p(\tilde{\mu}_j, \overline{\mu}_{j+1}) < \infty$, *with*

$$\Lambda_p(v, \sigma) := \sup_{\zeta \in (z_0, z_1)} v((z_0, \zeta]) \left\| \left(\frac{d\sigma}{ds}\right)^{-1} \right\|_{L^{1/(p-1)}([\zeta, z_1])} \qquad \text{if } 1 \leqslant p < \infty,$$

$$\Lambda_\infty(v, \sigma) := \operatorname*{ess\,sup}_{\zeta \in (z_0, z_1)} \frac{dv}{ds}(\zeta) \int_{\zeta}^{z_1} \left(\frac{d\sigma}{ds}\right)^{-1}(\xi)|d\xi|.$$

The Muckenhoupt inequality guarantees that if $f^{(j)} \in L^p(\mu_j)$ and $f^{(j+1)} \in L^p(\overline{\mu}_{j+1})$, then $f^{(j)} \in L^p(\overline{\mu}_j)$. If we work with absolutely continuous measures, we also say that a vectorial weight $\overline{w}$ is a completion of μ (or of w).

The following is an example of a completion when γ is an interval. It can be generalized to curves in an obvious way.

Example. We choose $\tilde{w}_j := 0$ if $\overline{w}_{j+1} \notin B_p((y, y+\varepsilon])$; if $\overline{w}_{j+1} \in B_p([y, y+\varepsilon])$ we set $\tilde{w}_j(x) := 1$ in $[y, y+\varepsilon]$; and if $\overline{w}_{j+1} \in B_p((y, y+\varepsilon]) \setminus B_p([y, y+\varepsilon])$ we take $\tilde{w}_j(x) := 1$ for $x \in [y+\varepsilon/2, y+\varepsilon]$, and

$$\tilde{w}_j(x) := \frac{d}{dx}\left\{ \left(\int_x^{y+\varepsilon} \overline{w}_{j+1}^{-1/(p-1)} \right)^{-p+1} \right\}$$
$$= \frac{(p-1)\overline{w}_{j+1}(x)^{-1/(p-1)}}{\left(\int_x^{y+\varepsilon} \overline{w}_{j+1}^{-1/(p-1)}\right)^p} \qquad \text{if } 1 < p < \infty,$$

$$\tilde{w}_j(x) := \|\overline{w}_{j+1}^{-1}\|^{-1}_{L^\infty([x,y+\varepsilon])} + \frac{d}{dx}\left(\|\overline{w}_{j+1}^{-1}\|^{-1}_{L^\infty([x,y+\varepsilon])}\right) \qquad \text{if } p = 1,$$

$$\tilde{w}_j(x) := \min\left\{ 1, \left(\int_x^{y+\varepsilon} \overline{w}_{j+1}^{-1} \right)^{-1} \right\} \qquad \text{if } p = \infty,$$

for $x \in (y, y+\varepsilon/2)$.

Remarks.

(1) We can define a left completion of μ with respect to z_0 in a similar way.

(2) If $\overline{w}_{j+1} \in B_p([z_0, z_1])$, then $\Lambda_p(\tilde{\mu}_j, \overline{w}_{j+1}) < \infty$ for any measure $\tilde{\mu}_j$ with $\tilde{\mu}_j((z_0, z_1]) < \infty$ if $1 \leqslant p < \infty$ and for any bounded weight $\tilde{w}_j$ if $p = \infty$. In particular, $\Lambda_p(1, \overline{w}_{j+1}) < \infty$.

(3) If μ, v are comparable measures, $\overline{v}$ is a right completion of v if and only if it is comparable to a right completion $\overline{\mu}$ of μ.

(4) If μ, v are two vectorial measures with the same absolutely continuous part, then $\overline{\mu}$ is a right completion of μ if and only if it is a right completion of v.

(5) If $\overline{\mu}$ is a right completion of μ with respect to z_0 in $(z_0, z_1]$ and $z_2 \in (z_0, z_1)$, the measure $\overline{\mu^*}$ defined by

$$\overline{\mu^*} = \begin{cases} \overline{\mu} & \text{in } [z_0, z_2], \\ \mu & \text{in } \gamma \setminus [z_0, z_2], \end{cases}$$

is a right completion of μ with respect to z_0 in $(z_0, z_2]$.

(6) If $\overline{\mu}$ is a right completion of μ with respect to z_0 in $(z_0, z_1]$ and $z_1 \in (z_0, z_2)$, $\overline{\mu}$ is also a right completion of μ with respect to z_0 in $(z_0, z_2]$ (it is enough to take $\tilde{\mu} \equiv 0$ in $(z_1, z_2]$).

(7) Let us fix $z_3 \in (z_0, z_1]$. If for every $z_2 \in (z_0, z_3]$ we have $\overline{w}_{j+1} \notin B_p((z_0, z_2])$, then there exists some $z_4 \in (z_0, z_1]$ such that every $\tilde{\mu}_j$ must be 0 in (z_0, z_4).

Definition 3.5. For $1 \leqslant p \leqslant \infty$ and a vectorial measure μ, we say that a point $z_0 \in \gamma$ is right j-regular (respectively, left j-regular), if there exist a right completion $\overline{\mu}$ (respectively, left completion) of μ in $[z_0, z_1]$ and $j < i \leqslant k$ such that $\overline{w}_i \in B_p([z_0, z_1])$ (respectively, $B_p([z_1, z_0])$). Also, we say that a point $z_0 \in \gamma$ is j-regular, if it is right and left j-regular.

Remarks.

(1) A point $z_0 \in \gamma$ is right j-regular (respectively, left j-regular), if at least one of the following properties is verified:

(a) There exist a right (respectively, left) neighbourhood $[z_0, z_1]$ (respectively, $[z_1, z_0]$) and $j < i \leqslant k$ such that $w_i \in B_p([z_0, z_1])$ (respectively, $B_p([z_1, z_0])$). Here we have chosen $\tilde{w}_j = 0$.

(b) There exist a right (respectively, left) neighbourhood $[z_0, z_1]$ (respectively, $[z_1, z_0]$) and $j < i \leqslant k$, $\alpha > 0$, $\delta < \delta_p$ with $\delta_p := (i - j)p - 1$ if $1 \leqslant p < \infty$ and $\delta_\infty := i - j - 1$, such that $w_i(z) \geqslant \alpha |z - z_0|^\delta$, for almost every $z \in [z_0, z_1]$ (respectively, $[z_1, z_0]$) and we have $|z - z_0| \asymp |\gamma^{-1}(z) - \gamma^{-1}(z_0)|$ in $[z_0, z_1]$ (respectively, $[z_1, z_0]$), when γ is the arc-length parametrization. See Lemma 3.4 in [RARP1].

(2) If z_0 is right j-regular (respectively, left), then it is also right i-regular (respectively, left) for each $0 \leqslant i \leqslant j$.

(3) We can take $i = j + 1$ in this definition since by the second remark after Definition 3.4 we can choose $\overline{w}_l = w_l + 1 \in B_p([z_0, z_1])$ for $j < l < i$, if $j + 1 < i$.

(4) If z_0 is right j-regular, by Remark 3 there exists a right completion $\overline{\mu}$ of μ with $\overline{w}_{j+1} \in B_p([z_0, z_1])$. If furthermore $w_k \in B_p((z_0, z_2])$ with $z_1 \in (z_0, z_2)$ we can assume that $\overline{w}_{j+1} \in B_p([z_0, z_2])$.

(5) If μ, v are two vectorial measures with the same absolutely continuous part, then z_0 is right j-regular (respectively, left) with respect to μ if and only if it is right j-regular (respectively, left) with respect to v.

When we use this definition we think of a point $\{z\}$ as the union of two half-points $\{z^+\}$ and $\{z^-\}$. With this convention, each one of the following sets:

$$(z_0, z_1) \cup (z_1, z_2) \cup \{z_1^+\} = (z_0, z_1) \cup [(z_1^+, z_2) \neq (z_0, z_2),$$

$$(z_0, z_1) \cup (z_1, z_2) \cup \{z_1^-\} = (z_0, z_1^-)] \cup (z_1, z_2) \neq (z_0, z_2),$$

has two connected components, and the set

$$(z_0, z_1) \cup (z_1, z_2) \cup \{z_1^-\} \cup \{z_1^+\} = (z_0, z_1) \cup (z_1, z_2) \cup \{z_1\} = (z_0, z_2),$$

is connected.

We only use this convention in order to study the sets of continuity of functions: we want that if $f \in C(A)$ and $f \in C(B)$, where A and B are union of arcs, then $f \in C(A \cup B)$. With the usual definition of continuity in an arc, if $f \in C([z_0, z_1)) \cap C([z_1, z_2])$ then we do not have $f \in C([z_0, z_2])$. Of course, we have $f \in C([z_0, z_2])$ if and only if $f \in C([z_0, z_1^-]) \cap C([z_1^+, z_2])$, where by definition, $C([z_1^+, z_2]) = C([z_1, z_2])$ and $C([z_0, z_1^-]) = C([z_0, z_1])$. This idea can be formalized with a suitable topological space.

Let us introduce some more notation. We denote by $\Omega^{(j)}$ the set of j-regular points or half-points, i.e., $z \in \Omega^{(j)}$ if and only if z is j-regular, we say that $z^+ \in \Omega^{(j)}$ if and only if z is right j-regular, and we say that $z^- \in \Omega^{(j)}$ if and only if z is left j-regular. Obviously, $\Omega^{(k)} = \emptyset$ and $\Omega_{j+1} \cup \cdots \cup \Omega_k \subseteq \Omega^{(j)}$. Observe that $\Omega^{(j)}$ depends on p (see Definition 3.5).

Remark. If $0 \leqslant j < k$ and J is an arc in γ, $J \subseteq \Omega^{(j)}$, then the set $J \setminus (\Omega_{j+1} \cup \cdots \cup \Omega_k)$ is discrete: If $z^+ \in J \setminus (\Omega_{j+1} \cup \cdots \cup \Omega_k)$, there exist $(z, z_1] \subseteq J$, a right completion $\overline{\mu}$ and $j < i \leqslant k$ with $\overline{w}_i \in B_p([z, z_1])$. Then there exist $z_2 \in (z, z_1]$ and $i \leqslant l \leqslant k$ with $w_l \in B_p((z, z_2])$ and consequently $(z, z_2) \subseteq \Omega_{j+1} \cup \cdots \cup \Omega_k$ (see Remark 7 to Definition 3.4). The same is true for z^- with the obvious changes.

DEFINITION 3.6. We say that the vectorial measure $\mu = (\mu_0, \ldots, \mu_k)$ is p-admissible if

$$(\mu_j - (w_j)|_{\Omega_j})(\gamma \setminus \Omega^{(j)}) = 0 \qquad \text{for } 1 \leqslant j \leqslant k.$$

We say that μ is strongly p-admissible if $\operatorname{supp}(\mu_j - (w_j)|_{\Omega_j}) \subseteq \Omega^{(j)}$, for $1 \leqslant j \leqslant k$.

We use the letter p in p-admissible in order to emphasize the dependence on p (recall that $\Omega^{(j)}$ depends on p).

Remarks

(1) There is no condition on μ_0.

(2) We have $(\mu_k)_s \equiv 0$ and $w_k = 0$ in almost every $z \in \gamma \setminus \Omega_k$, since $\Omega^{(k)} = \emptyset$.

(3) Every absolutely continuous measure w with $w_j(z) = 0$ in almost every $z \in \gamma \setminus \Omega_j$ for $1 \leqslant j \leqslant k$ is p-admissible.

(4) Recall that we are identifying w_j with the measure $w_j\, ds$.

(5) This definition is more general than the definition of p-admissible measure in [RARP1]; there we always assume $w_j(z) = 0$ in $\gamma \setminus \Omega_j$. There exist weights which do not satisfy this reasonable condition: Consider a Cantor set C in $[0, 1]$ with positive length and define $w_1 := 1$ in C and $w_1(x) := \operatorname{dist}(x, C)$ if $x \in \mathbf{R} \setminus C$; it is clear that $\Omega_1 = \mathbf{R} \setminus C$ and $w_1 = 1$ in C.

DEFINITION 3.7 (Sobolev Space). Let us consider $1 \leqslant p \leqslant \infty$ and $\mu = (\mu_0, \dots, \mu_k)$ a p-admissible vectorial measure. We define the Sobolev space $W^{k,p}(\gamma, \mu)$ as the space of equivalence classes of

$$V^{k,p}(\gamma, v) := \{ f : \gamma \to \mathbf{C} | f^{(j)} \in AC^1_{\mathrm{loc}}(\Omega^{(j)}) \quad \text{for } 0 \leqslant j < k \text{ and}$$
$$||f^{(j)}||_{L^p(\gamma,\mu_j)} < \infty \text{ for } 0 \leqslant j \leqslant k \},$$

with respect to the seminorm

$$||f||_{W^{k,p}(\gamma,\mu)} := \left(\sum_{j=0}^{k} ||f^{(j)}||^p_{L^p(\gamma,\mu_j)} \right)^{1/p} \qquad \textit{for } 1 \leqslant p < \infty$$

and

$$||f||_{W^{k,\infty}(\gamma,\mu)} := \max_{0 \leqslant j \leqslant k} ||f^{(j)}||_{L^\infty(\gamma,\mu_j)},$$

where

$$||g||_{L^\infty(\gamma,\mu_j)} := \max \left\{ \operatorname*{ess\,sup}_{z \in \gamma} |g(z)| w_j(z), \sup_{z \in \mathrm{supp}\,(\mu_j)_s} |g(z)| \right\}$$

and we assume the usual convention $\sup \emptyset = -\infty$.

Remark. It is natural to ask for $f^{(j)} \in AC^1_{\mathrm{loc}}(\Omega^{(j)})$, since when the $(\mu_j)_s$-measure of the set where $|f^{(j)}|$ is not continuous is positive, the integral $\int |f^{(j)}|^p \, d(\mu_j)_s$ does not make sense.

4. TECHNICAL RESULTS

LEMMA 4.1. *Let $1 \leqslant p \leqslant \infty$, $I = [z_0, z_1]$ a compact arc in γ and $\mu = (\mu_0, \dots, \mu_k)$ a p-admissible vectorial measure in I,* with *$(z_0, z_1] \subseteq \Omega^{(k_0-1)}$ for some $0 < k_0 \leqslant k$. If we construct a right completion $\overline{\mu}$ of μ with respect to the point z_0, satisfying $\overline{\mu}_j = \mu_j$ for $k_0 \leqslant j \leqslant k$, then there exists a positive constant c such that*

$$c||g^{(j)}||_{L^p(I,\bar{\mu}_j)} \leqslant \sum_{i=j}^{k_0} ||g^{(i)}||_{L^p(I,\mu_i)} + \sum_{i=j}^{k_0-1} |g^{(i)}(z_1)|,$$

for all $0 \leqslant j < k_0$ and $g \in V^{k,p}(I, \mu)$. In particular, we have

$$c||g||_{W^{k,p}(I,\bar{\mu})} \leqslant ||g||_{W^{k,p}(I,\mu)} + \sum_{j=0}^{k_0-1} |g^{(j)}(z_1)|,$$

for all $g \in V^{k,p}(I, \mu)$.

Proof. The fact $\bar{\mu}_j = \mu_j$ for $k_0 \leqslant j \leqslant k$ and the first inequality give the second one, Then we only need to prove the first inequality. If $g \in V^{k,p}(I, \mu)$, we have $g^{(j)} \in AC^1_{\text{loc}}((z_0, z_1))$ since $(z_0, z_1] \subseteq \Omega^{(j)} \subseteq \Omega^{(k_0-1)}$. Muckenhoupt inequality gives

$$c||g^{(j)}(z) - g^{(j)}(z_1)||_{L^p(I,\tilde{\mu}_j)} \leqslant ||g^{(j+1)}||_{L^p(I,\overline{\mu}_{j+1})}$$

for $0 \leqslant j < k_0$ (we can consider the point z_1 as the limit of the completion by Remarks 5 and 6 to Definition 3.4). Then we have for $1 \leqslant p \leqslant \infty$,

$$c||g^{(j)}||_{L^p(I,\tilde{\mu}_j)} \leqslant ||g||^{(j+1)}||_{L^p(I,\overline{\mu}_{j+1})} + |g^{(j)}(z_1)|$$

since $\tilde{\mu}_j(I) < \infty$ if $1 \leqslant p < \infty$, and $\tilde{w}_j \in L^\infty(I)$ and $\tilde{\mu}_j$ is absolutely continuous if $p = \infty$. This inequality gives now

$$c||g^{(j)}||_{L^p(I,\overline{\mu}_j)} \leqslant ||g^{(j)}||_{L^p(I,\mu_j)} + ||g^{(j+1)}||_{L^p(I,\overline{\mu}_{j+1})} + |g^{(j)}(z_1)|$$

for $0 \leqslant j < k_0$. This fact and $\overline{\mu}_{k_0} = \mu_{k_0}$ prove the first inequality. ∎

LEMMA 4.2. *Let us consider $1 \leqslant p \leqslant \infty$, I a compact arc in γ, $z_0 \in \operatorname{int} I$ and μ_k an absolutely continuous measure in I, with $w_k \in B_p(\operatorname{int} I)$. Assume also that $\Omega^{(0)} = I$ for $\mu = (0, \ldots, 0, \mu_k)$. Then, there exists a positive constant c with*

$$\left|\left|\int_{z_0}^{z} \frac{g^{(k)}(\zeta)}{(k-1)!} (z-\zeta)^{k-1}\, d\zeta\right|\right|_{L^\infty(I)} \leqslant c||g^{(k)}||_{L^p(I,\mu_k)}$$
for every $g \in AC^k_{\text{loc}}(\operatorname{int} I)$.

Furthermore, if $I_j \subseteq \Omega^{(j)}$ is a compact arc $(0 \leqslant j < k)$, then there exists a positive constant c with

$$\left|\left|\int_{z_0}^{z} \frac{g^{(k)}(\zeta)}{(k-j-1)!} (z-\zeta)^{k-j-1}\, d\zeta\right|\right|_{L^\infty(I_j)} \leqslant c||g^{(k)}||_{L^p(I,\mu_k)},$$
for every $g \in AC^k_{\text{loc}}(\operatorname{int} I)$.

Proof. We prove the first inequality; the second one is an immediate consequence of it. Without loss of generality we can assume that $g \in V^{k,p}(I, \mu)$, since otherwise $||g^{(k)}||_{L^p(I,\mu_k)} = \infty$, and the inequality is trivial. Assume that $I = [z_1, z_2]$. Since z_1 is right 0-regular, by

Remark 4 to Definition 3.5, there exists a right completion $\overline{\mu}$ of μ with respect to z_1, with $\overline{w}_1 \in B_p([z_1, z_0])$. Then, by Lemma 4.1 we have that

$$c||g'||_{L^p([z_1,z_0],\overline{\mu}_1)} \leqslant ||g||_{W^{k,p}([z_1,z_0],\mu)} + \sum_{j=1}^{k-1} |g^{(j)}(z_0)|$$

for all $g \in V^{k,p}([z_1, z_0], \mu)$, and so

$$\begin{aligned} ||g||_{L^\infty([z_1,z_0])} &\leqslant ||g'||_{L^1([z_1,z_0])} + |g(z_0)| \leqslant c||g'||_{L^p([z_1,z_0],\overline{\mu}_1)} + |g(z_0)| \\ &\leqslant c||g||_{W^{k,p}([z_1,z_0],\mu)} + c\sum_{j=0}^{k-1} |g^{(j)}(z_0)|. \end{aligned}$$

A symmetric argument gives

$$||g||_{L^\infty([z_0,z_2])} \leqslant c||g||_{W^{k,p}([z_0,z_2],\mu)} + c\sum_{j=0}^{k-1} |g^{(j)}(z_0)|.$$

Since the function

$$a(z) := \int_{z_0}^{z} \frac{g^{(k)}(\zeta)}{(k-1)!} (z-\zeta)^{k-1}\, d\zeta$$

verifies $a^{(j)}(z_0) = 0$ for $0 \leqslant j < k$, the proof is finished. ∎

PROPOSITION 4.1. *Let us consider* $1 \leqslant p \leqslant \infty$, *$I$ a compact arc in γ and* $\mu = (\mu_0, 0, \ldots, 0, \mu_k)$ *a vectorial measure in I with μ_k absolutely continuous,* $w_k \in B_p(\operatorname{int} I)$ *and* $\#\operatorname{supp}(\mu_0|_{\Omega^{(0)}\cap I}) \geqslant k$. *Define*

$$X = \{f \in AC^k_{\mathrm{loc}}(I) : ||f^{(k)}||_{L^p(I,\mu_k)} < \infty\}.$$

Then, given compact arcs $I_j \subseteq \Omega^{(j)} \cap I$ for $0 \leqslant j < k$, there exists a positive constant c with

$$c\sum_{j=0}^{k-1} ||f^{(j)}||_{L^\infty(I_j)} \leqslant ||f||_{L^p(I,\mu_0)} + ||f^{(k)}||_{L^p(I,\mu_k)} \qquad \textit{for every } f \in X.$$

Proof. Without loss of generality we can assume that $\mu_0(I) < \infty$, since in other case the right-hand side of the inequality is greater. Without loss of generality we can assume that $\Omega^{(0)} \cap I = I$, since otherwise we can change I by $I_0 \cup I_1 \cup \cdots \cup I_{k-1} \cup \Lambda$, where Λ is any compact arc contained in $\Omega^{(0)} \cap I$ and with $\#\operatorname{supp}(\mu_0|_\Lambda) \geqslant k$. We can assume also $\emptyset \neq I_{k-1} \subseteq I_{k-2} \subseteq \cdots \subseteq I_0 = I$ and even $I_j = I$ if $\Omega^{(j)} \cap I = I$.

We prove first that the normed spaces $(X, ||\cdot||_A)$ and $(X, ||\cdot||_B)$ are complete, where

$$||f||_A := ||f||_{L^p(I,\mu_0)} + ||f^{(k)}||_{L^p(I,\mu_k)}, \quad ||f||_B := \sum_{j=0}^{k-1} ||f^{(j)}||_{L^\infty(I_j)} + ||f^{(k)}||_{L^p(I,\mu_k)}.$$

We start now by proving the completeness of the space $(X, ||\cdot||_A)$. Observe first that $||\cdot||_A$ is a norm in X: if $||f^k||_{L^p(I,\mu_k)} = 0$, then $f \in P_{k-1}$; this fact and $||f||_{L^p(I,\mu_0)} = 0$ gives $f = 0$ in I, since $||\cdot||_{L^p(I,\mu_0)}$ is a norm on P_{k-1} (recall that μ_0 is finite and $\#\mathrm{supp}(\mu_0|_{\Omega^{(0)}\cap I}) \geqslant k$). Let us consider a Cauchy sequence $\{f_n\} \subset (X, ||\cdot||_A)$. Each function can be written as

$$f_n(z) = \sum_{j=0}^{k-1} \frac{f_n^{(j)}(z_0)}{j!}(z - z_0)^j + \int_{z_0}^{z} \frac{f_n^{(k)}(\zeta)}{(k-1)!}(z-\zeta)^{k-1}\, d\zeta,$$

with $z_0 \in I_{k-1}$. So,

$$f_n(z) - f_m(z) = \sum_{j=0}^{k-1} \frac{f_n^{(j)}(z_0) - f_m^{(j)}(z_0)}{j!}(z - z_0)^j + \int_{z_0}^{z} \frac{f_n^{(k)}(\zeta) - f_m^{(k)}(\zeta)}{(k-1)!}(z-\zeta)^{k-1}\, d\zeta.$$

Lemma 4.2 gives

$$\begin{aligned} &\left|\left|\int_{z_0}^{z} \frac{f_n^{(k)}(\zeta) - f_m^{(k)}(\zeta)}{(k-1)!}(z-\zeta)^{k-1}\, d\zeta\right|\right|_{L^p(I,\mu_0)} \\ &\quad \leqslant c\left|\left|\int_{z_0}^{z} \frac{f_n^{(k)}(\zeta) - f_m^{(k)}(\zeta)}{(k-1)!}(z-\zeta)^{k-1}\, d\zeta\right|\right|_{L^\infty(I)} \\ &\quad \leqslant c||f_n^{(k)} - f_m^{(k)}||_{L^p(I,\mu_k)} \to 0, \end{aligned}$$

as $n, m \to \infty$, since μ_0 is finite. Using that $||f_n - f_m||_{L^p(I,\mu_0)} \to 0$ as $n, m \to \infty$ we obtain

$$\left|\left|\sum_{j=0}^{k-1} \frac{f_n^{(j)}(z_0) - f_m^{(j)}(z_0)}{j!}(z - z_0)^j\right|\right|_{L^p(I,\mu_0)} \to 0,$$

as $n, m \to \infty$. Since $||\cdot||_{L^p(I,\mu_0)}$ is a norm on P_{k-1}, we have that $f_n^{(j)}(z_0) \to c_j$ for some constants c_j, with $0 \leqslant j \leqslant k-1$ and

$$\left|\left|\sum_{j=0}^{k-1} \frac{f_n^{(j)}(z_0) - c_j}{j!}(z - z_0)^j\right|\right|_{L^p(I,\mu_0)} \to 0, \tag{4.1}$$

as $n \to \infty$. We obviously have functions $F_0 \in L^p(I, \mu_0)$, $F_k \in L^p(I, \mu_k)$ such that

$$||F_0 - f_n||_{L^p(I,\mu_0)} + ||F_k - f_n^{(k)}||_{L^p(I,\mu_k)} \to 0,$$

as $n \to \infty$. Now, we can define

$$\tilde{F}_0(z) = \sum_{j=0}^{k-1} \frac{c_j}{j!}(z - z_0)^j + \int_{z_0}^{z} \frac{F_k(\zeta)}{(k-1)!}(z-\zeta)^{k-1}\, d\zeta.$$

Next we prove $\tilde{F}_0 = F_0$, μ_0-almost everywhere in I; this fact gives $||\tilde{F}_0 - f_n||_A \to 0$ as $n \to \infty$. We have this by (4.1) and

$$\left|\left| \int_{z_0}^{z} \frac{f_n^{(k)}(\zeta) - F_k(\zeta)}{(k-1)!}(z-\zeta)^{k-1}\, d\zeta \right|\right|_{L^p(I,\mu_0)} \leqslant c||f_n^{(k)} - F_k||_{L^p(I,\mu_k)} \to 0,$$

as $n \to \infty$. This gives the completeness of the space $(X, ||\cdot||_A)$.

We prove now the completeness of the space $(X, ||\cdot||_B)$. Let us consider a Cauchy sequence $\{f_n\} \subset (X, ||\cdot||_B)$. For each $0 \leqslant j \leqslant k$ there exists F_j with

$$||F_j - f_n^{(j)}||_{L^\infty(I_j)} \to 0 \qquad \text{for } 0 \leqslant j < k, \qquad ||F_k - f_n^{(k)}||_{L^p(I,\mu_k)} \to 0,$$

as $n \to \infty$. If we fix $z_0 \in I_{k-1}$, we have

$$f_n^{(j)}(z) = \sum_{i=j}^{k-1} \frac{f_n^{(i)}(z_0)}{(i-j)!}(z-z_0)^{i-j} + \int_{z_0}^{z} \frac{f_n^{(k)}(\zeta)}{(k-j-1)!}(z-\zeta)^{k-j-1}\, d\zeta$$

for $z \in I_j$ and $0 \leqslant j < k$. By Lemma 4.2 and the uniform convergence of $f_n^{(j)}$ in I_j, we have

$$F_j(z) = \sum_{i=j}^{k-1} \frac{F_i(z_0)}{(i-j)!}(z-z_0)^{i-j} + \int_{z_0}^{z} \frac{F_k(\zeta)}{(k-j-1)!}(z-\zeta)^{k-j-1}\, d\zeta$$

for $z \in I_j$ and $0 \leqslant j < k$. Consequently $F_0^{(j)} = F_j$ in I_j, for $0 \leqslant j < k$ and $F_0^{(k)} = F_k$ in I. This gives the completeness of $(X, ||\cdot||_B)$.

Observe that $||f||_A \leqslant c||f||_B$ for every $f \in X$. Since $(X, ||\cdot||_A)$ and $(X, ||\cdot||_B)$ are Banach spaces, the open mapping theorem in Banach spaces gives $||f||_B \leqslant c||f||_A$ for every $f \in X$, and this finishes the proof. ∎

PROPOSITION 4.2. *Let us consider* $1 \leqslant p \leqslant \infty$, *$I$ a compact arc in γ and* $\mu = (\mu_0, \dots, \mu_k)$ *a p-admissible vectorial measure in I, with* $w_k \in B_p(\operatorname{int} I)$, *and* $\#\operatorname{supp}(\mu_0|_{\Omega^{(0)} \cap I}) \geqslant k$. *Then, given compact arcs* $I_j \subseteq \Omega^{(j)} \cap I$ *for* $0 \leqslant j < k$, *there exists a positive constant c with*

$$c\sum_{j=0}^{k-1}\|f^{(j)}\|_{L^\infty(I_j)}\leqslant\|f\|_{W^{k,p}(I,\mu)}\qquad \text{for every } f\in V^{k,p}(I,\mu).$$

Remark. Observe that in Proposition 4.1 the set $\Omega^{(j)}$ only depends on w_k. However, in Proposition 4.2 the set $\Omega^{(j)}$ depends on $w_{j+1},\dots,w_k$.

Proof. By Proposition 4.1 the result holds if $I_j\subset\operatorname{int}I$. Therefore, we only need to obtain the bounds in a neighbourhood of ∂I. If $I=[z_1,z_2]$, assume that $z_1\in I_j$ for some $0\leqslant j<k$ (the case $z_2\in I_j$ is symmetric). Since $I_j\subset\Omega^{(j)}\cap I$, there exist a completion $\overline{\mu}$ and $z_0\in(z_1,z_2)$ with $\overline{w}_{j+1}\in B_p([z_1,z_0])$. Then Lemma 4.1 and Proposition 4.1 give

$$\begin{aligned}\|f^{(j)}\|_{L^\infty([z_1,z_0])}&\leqslant\|f^{(j+1)}\|_{L^1([z_1,z_0])}+|f^{(j)}(z_0)|\\&\leqslant c\|f^{(j+1)}\|_{L^p([z_1,z_0],\overline{\mu}_{j+1})}+|f^{(j)}(z_0)|\\&\leqslant c\|f\|_{W^{k,p}([z_1,z_0],\mu)}+c\sum_{i=0}^{k-1}|f^{(i)}(z_0)|\leqslant c\|f\|_{W^{k,p}(I,\mu)}.\qquad\blacksquare\end{aligned}$$

DEFINITION 4.1. Let us consider $1\leqslant p\leqslant\infty$ and μ a p-admissible vectorial measure in γ. Let us define the space $\mathscr{K}(\gamma,\mu)$ as

$$\mathscr{K}(\gamma,\mu):=\{g:\Omega^{(0)}\to\mathbf{C}/g\in V^{k,p}(\gamma,\mu|_{\Omega^{(0)}}),\|g\|_{W^{k,p}(\gamma,\mu|_{\Omega^{(0)}})}=0\}.$$

$\mathscr{K}(\gamma,\mu)$ is the equivalence class of 0 in $W^{k,p}(\gamma,\mu|_{\Omega^{(0)}})$. Therefore, $\|\cdot\|_{W^{k,p}(\gamma,\mu)}$ is a norm if and only if $\mathscr{K}(\gamma,\mu)=\{0\}$. It plays an important role in the study of the multiplication operator in Sobolev spaces (see Theorem 8.3 below) and in the following definition of classes $\mathscr{C}$ and $\mathscr{C}_0$, which will be crucial in the study of Sobolev spaces (see Theorems 4.1, 4.2 and 5.1 below).

The case in which $\|\cdot\|_{W^{k,p}(\gamma,\mu)}$ is a norm is the most interesting. However we need something more in order to prove part (a) of Theorem 4.1 below: this additional condition is what we present in our following definition of class $\mathscr{C}_0$. Roughly speaking, $\mu\in\mathscr{C}_0$ if $\|\cdot\|_{W^{k,p}(M_n,\mu)}$ is a norm for some sequence of compact sets $\{M_n\}$ growing to γ. This condition is exactly what we need since in the proof of Theorem 4.1 we approximate γ by compact sets.

If $\mu\notin\mathscr{C}_0$ we still can prove part (b) of Theorem 4.1 by adding some Dirac deltas to μ_0; we only add the exact amount that we need. This leads to the definition of class $\mathscr{C}$.

DEFINITION 4.2. Let us consider $1\leqslant p\leqslant\infty$ and μ a p-admissible vectorial measure in γ. We say (γ,μ) belongs to the class $\mathscr{C}_0$ if there exist compact sets M_n, which are a finite union of compact arcs in γ, such that

(i) M_n intersects at most a finite number of connected components of $\Omega_1 \cup \cdots \cup \Omega_k$,

(ii) $\mathcal{K}(M_n, \mu) = \{0\}$,

(iii) $M_n \subseteq M_{n+1}$,

(iv) $\bigcup_n M_n = \Omega^{(0)}$.

We say that (γ, μ) belongs to the class $\mathcal{C}$ if there exists a measure $\mu_0' = \mu_0 + \sum_{m \in D} c_m \delta_{z_m}$ with $c_m > 0$, $\{z_m\} \subset \Omega^{(0)}$, $D \subseteq \mathbf{N}$ and $(\gamma, \mu') \in \mathcal{C}_0$, where $\mu' = (\mu_0', \mu_1, \ldots, \mu_k)$ is minimal in the following sense: there exists $\{M_n\}$ corresponding to $(\gamma, \mu') \in \mathcal{C}_0$ such that if $\mu_0'' = \mu_0' - c_{m_0} \delta_{z_{m_0}}$ with $m_0 \in D$ and $\mu'' = (\mu_0'', \mu_1, \ldots, \mu_k)$, then $\mathcal{K}(M_n, \mu'') \neq \{0\}$ if $z_{m_0} \in M_n$.

Remarks

(1) The condition $(\gamma, \mu) \in \mathcal{C}$ is not very restrictive. In fact, the proof of Theorem 4.1 gives that if $\Omega^{(0)} \setminus (\Omega_1 \cup \cdots \cup \Omega_k)$ has only a finite number of points in each connected component of $\Omega^{(0)}$, then $(\gamma, \mu) \in \mathcal{C}$. If furthermore $\mathcal{K}(\gamma, \mu) = \{0\}$, we have $(\gamma, \mu) \in \mathcal{C}_0$.

(2) Since the restriction of a function of $\mathcal{K}(\gamma, \mu)$ to M_n is in $\mathcal{K}(M_n, \mu)$ for every n, then $(\gamma, \mu) \in \mathcal{C}_0$ implies $\mathcal{K}(\gamma, \mu) = \{0\}$.

(3) If $(\gamma, \mu) \in \mathcal{C}_0$, then $(\gamma, \mu) \in \mathcal{C}$, with $\mu' = \mu$.

(4) The proof of Theorem 4.1 gives that if for every connected component Λ of $\Omega_1 \cup \cdots \cup \Omega_k$ we have $\mathcal{K}(\overline{\Lambda}, \mu) = \{0\}$, then $(\gamma, \mu) \in \mathcal{C}_0$. The Condition $\#\mathrm{supp}\, \mu_0|_{\overline{\Lambda} \cap \Omega^{(0)}} \geqslant k$ implies $\mathcal{K}(\overline{\Lambda}, \mu) = \{0\}$.

The next results play a central role in the theory of Sobolev spaces in curves. The first one answers to the following main question: when the evaluation functional of f (or $f^{(j)}$) in a point is a bounded operator in $W^{k,p}(\gamma, \mu)$?

THEOREM 4.1. *Let us consider* $1 \leqslant p \leqslant \infty$ *and* $\mu = (\mu_0, \ldots \mu_k)$ *a* p*-admissible vectorial measure in* γ. *Let* K_j *be a finite union of compact arcs contained in* $\Omega^{(j)}$, *for* $0 \leqslant j < k$ and $\overline{\mu}$ *a right (or left) completion of* μ. *Then*:

(a) *If* $(\gamma, \mu) \in \mathcal{C}_0$ *there exist positive constants* $c_1 = c_1(K_0, \ldots, K_{k-1})$ *and* $c_2 = c_2(\overline{\mu}, K_0, \ldots, K_{k-1})$ *such that*

$$c_1 \sum_{j=0}^{k-1} \|g^{(j)}\|_{L^\infty(K_j)} \leqslant \|g\|_{W^{k,p}(\gamma,\mu)}, \qquad c_2 \|g\|_{W^{k,p}(\gamma,\overline{\mu})} \leqslant \|g\|_{W^{k,p}(\gamma,\mu)},$$
$$\forall g \in V^{k,p}(\gamma, \mu).$$

(b) *If* $(\gamma, \mu) \in \mathscr{C}$ *there exist positive constants* $c_3 = c_3(K_0, \dots, K_{k-1})$ *and* $c_4 = c_4(\overline{\mu}, K_0, \dots, K_{k-1})$ *such that for every* $g \in V^{k,p}(\gamma, \mu)$, *there exists* $g_0 \in V^{k,p}(\gamma, \mu)$, *independent of* $K_0, \dots, K_{k-1}, c_3, c_4$ *and* $\overline{\mu}$, *with*

$$||g_0 - g||_{W^{k,p}(\gamma,\mu)} = 0,$$

$$c_3 \sum_{j=0}^{k-1} ||g_0^{(j)}||_{L^\infty(K_j)} \leqslant ||g_0||_{W^{k,p}(\gamma,\mu)} = ||g||_{W^{k,p}(\gamma,\mu)},$$

$$c_4 ||g_0||_{W^{k,p}(\gamma,)\overline{\mu}} \leqslant ||g||_{W^{k,p}(\gamma,\mu)}.$$

Furthermore, if g_0, f_0 *are, respectively, these representatives of* g, f, *we have with the same constants* c_3, c_4

$$c_3 \sum_{j=0}^{k-1} ||g_0^{(j)} - f_0^{(j)}||_{L^\infty(K_j)} \leqslant ||g - f||_{W^{k,p}(\gamma,\mu)},$$

$$c_4 ||g_0 - f_0||_{W^{k,p}(\gamma,\overline{\mu})} \leqslant ||g - f||_{W^{k,p}(\gamma,\mu)}.$$

Proof. The main ingredient in the proof is Proposition 4.2. We only need to cut in an appropriate way the compact sets K_j in order to fulfill the hypotheses of this proposition. To see the details we can follow the argument in the proof of Theorem 4.3 in [RARP1] (Proposition 4.2 plays the role of Corollary 4.1 in [RARP1]). ∎

THEOREM 4.2. *Let us consider* $1 \leqslant p \leqslant \infty$ *and* μ *a* p-*admissible vectorial measure in* γ. *Let* K_j *be a finite union of compact arcs contained in* $\Omega^{(j)}$, *for* $0 \leqslant j < k$. *Then*:

(a) *If* $(\gamma, \mu) \in \mathscr{C}_0$ *there exists a positive constant* $c_1 = c_1(K_0, \dots, K_{k-1})$ *such that*

$$c_1 \sum_{j=0}^{k-1} ||g^{(j+1)}||_{L^1(K_j)} \leqslant ||g||_{W^{k,p}(\gamma,\mu)}, \qquad \forall g \in V^{k,p}(\gamma, \mu).$$

(b) *If* $(\gamma, \mu) \in \mathscr{C}$ *there exists a positive constant* $c_2 = c_2(K_0, \dots, K_{k-1})$ *such that for every* $g \in V^{k,p}(\gamma, \mu)$, *there exists* $g_0 \in V^{k,p}(\gamma, \mu)$, (*the same function as in Theorem* 4.1), *with*

$$||g_0 - g||_{W^{k,p}(\gamma,\mu)} = 0,$$

$$c_2 \sum_{j=0}^{k-1} ||g_0^{(j+1)}||_{L^1(K_j)} \leqslant ||g_0||_{W^{k,p}(\gamma,\mu)} = ||g||_{W^{k,p}(\gamma,\mu)}.$$

Furthermore, if g_0, f_0, *are, respectively, these representatives of* g, f, *we have with the same constant* c_2

$$c_2 \sum_{j=0}^{k-1} ||g_0^{(j+1)} - f_0^{(j+1)}||_{L^1(K_j)} \leqslant ||g - f||_{W^{k,p}(\gamma,\mu)}.$$

The representatives g_0, f_0 *are the same as in Theorem* 4.1.

Proof. We only prove part (b) since (a) is simpler. Given a function $g \in V^{k,p}(\gamma, \mu)$, let us choose g_0 as in Theorem 4.1(b). Fix $0 \leqslant j < k$. Since $K_j \subseteq \Omega^{(j)}$, given any point $z \in K_j$, there exist an arc J_z and a completion $\overline{w}^z$ of w with $z \in J_z$ and $\overline{w}_{j+1}^z \in B_p(J_z)$. The compactness of K_j gives that there exists a finite set of points $z_1, \ldots, z_l$ with $K_j \subseteq J_{z_1} \cup \cdots \cup J_{z_l}$.

If we define $w_{j+1}^* := \sum_{i=1}^{l} \overline{w}_{j+1}^{z_i} \chi_{J_{z_i}}$ the second inequality in Theorem 4.1(b) gives

$$c||g_0^{(j+1)}||_{L^p(K_j, w_{j+1}^*)} \leqslant ||g_0||_{W^{k,p}(\gamma,\mu)},$$

and this finishes the proof of the first inequality, since $w_{j+1}^* \in B_p(K_j)$. The proof of the second one is similar. ∎

A simple modification in the proof of Theorem 4.2 gives Corollary 4.1. We use the symbol $W^{k-m,p}(\gamma, \mu)$ to denote the Sobolev space $W^{k-m,p}(\gamma, (\mu_m, \ldots, \mu_k))$.

Corollary 4.1. *Let us consider* $1 \leqslant p \leqslant \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a* p*-admissible vectorial measure in* γ. *For some* $0 < m \leqslant k$, *assume that* $(\gamma, (\mu_m, \ldots, \mu_k)) \in \mathscr{C}_0$. *Let* K *be a finite union of compact intervals contained in* $\Omega^{(m-1)}$. *Then there exists a positive constant* $c_1 = c_1(K)$ *such that*

$$c_1 ||g||_{L^1(K)} \leqslant ||g||_{W^{k-m,p}(\gamma,\mu)}, \qquad \forall g \in V^{k-m,p}(\gamma, \mu).$$

5. COMPLETENESS

Theorem 5.1. *Let us consider* $1 \leqslant p \leqslant \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a* p*-admissible vectorial measure in* γ *with* $(\gamma, \mu) \in \mathscr{C}$. *Then the Sobolev space* $w^{k,p}(\gamma, \mu)$ *is complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $W^{k,p}(\gamma, \mu)$. For each n, let us choose a representative of the class of $f_n \in W^{k,p}(\gamma, \mu)$ (which we also denote by f_n) as in Theorems 4.1 and 4.2. Therefore, for each $0 \leqslant j \leqslant k$, $\{f_n^{(j)}\}$ is a Cauchy sequence in $L^p(\gamma, \mu_j)$ and it converges to a function $g_j \in L^p(\gamma, \mu_j)$.

We only need to prove that, for each $0 \leqslant j \leqslant k-1$, g_j is (perhaps modified in a set of zero μ_j-measure) a function belonging to $AC^1_{\mathrm{loc}}(\Omega^{(j)})$ such that $g_j' = g_{j+1}$ in $\Omega^{(j)}$.

Let us consider any compact arc $K \subseteq \Omega^{(j)}$ (K can be the whole curve γ if $\Omega^{(j)} = \gamma$ and it is a compact curve). By Theorems 4.1(b) and 4.2(b) we know that there exists a positive constant c such that for every $n, m \in \mathbf{N}$

$$||f_n^{(j)} - f_m^{(j)}||_{L^\infty(K)} + ||f_n^{(j+1)} - f_m^{(j+1)}||_{L^1(K)} \leqslant c \sum_{i=0}^{k} ||f_n^{(i)} - f_m^{(i)}||_{L^p(\gamma,\mu_i)}.$$

As $\{f_n^{(j)}\} \subset C(K)$, there exists a function $h_j \in C(K)$ such that

$$c||f_n^{(j)} - h_j||_{L^\infty(K)} \leqslant \sum_{i=0}^{k} ||f_n^{(i)} - g_i||_{L^p(\gamma,\mu_i)}.$$

Since we can take as K any compact arc contained in $\Omega^{(j)}$, we obtain that the function h_j can be extended to $\Omega^{(j)}$ and we have in fact $h_j \in C(\Omega^{(j)})$. It is obvious that $g_j = h_j$ in $\Omega^{(j)}$ (except for at most a set of zero μ_j-measure), since $f_n^{(j)}$ converges to g_j in the norm of $L^p(\gamma, \mu_j)$ and to h_j uniformly on each compact arc $K \subseteq \Omega^{(j)}$. Therefore we can assume that $g_j \in C(\Omega^{(j)})$.

Let us see now that $g_j' = g_{j+1}$ in K . We have for $z, z_0 \in K$ that

$$f_n^{(j)}(z) = f_n^{(j)}(z_0) + \int_{z_0}^{z} f_n^{(j+1)}(\zeta)\, d\zeta.$$

The uniform convergence of $f_n^{(j)}$ in K and the L^1-convergence of $f_n^{(j+1)}$ in K give that

$$g_j(z) = g_j(z_0) + \int_{z_0}^{z} g_{j+1}(\zeta)\, d\zeta. \quad \blacksquare$$

DEFINITION 5.1. A vectorial measure $\mu = (\mu_0, \ldots, \mu_k)$ in the complex plane belongs to (ESD) extended sequentially dominated if there exists a positive constant c such that $\mu_{j+i} \leqslant c\mu_j$ for $0 \leqslant j < k$.

Remark. If $\mu \in \mathrm{ESD}$ is a p-admissible vectorial measure in a curve γ, then $(\gamma, \mu) \in \mathscr{C}_0$ (see Remark 4 to Definition 4.2). A vectorial measure μ is sequentially dominated if and only if $\mu \in \mathrm{ESD}$ and $\#\mathrm{supp}\, \mu_0 = \infty$. If $\mu \in \mathrm{ESD}$, 0 is the unique polynomial q with $||q||_{W^{k,p}(\mathbf{C},\mu)} = 0$ if and only if $\#$ $\mathrm{supp}\, \mu_0 = \infty$.

THEOREM 5.2. *Let us consider $1 \leqslant p \leqslant \infty$, $\gamma : I \to \mathbf{C}$ a curve with $\gamma' \in W^{k-1,\infty}(I)$, and $\mu = (\mu_0, \ldots, \mu_k)$ a p-admissible vectorial measure in γ with $(\gamma, \mu) \in \mathscr{C}$. Let us assume that $\mu \in \mathrm{ESD}$ if $k \geqslant 2$. Then there exists a p-admissible vectorial measure μ^* in I, with $(I, \mu^*) \in \mathscr{C}$, and $\mu^* \in \mathrm{ESD}$ if*

$k \geqslant 2$, such that the spaces $W^{k,p}(I, \mu^*)$ and $W^{k,p}(\gamma, \mu)$ are isomorphic as normed spaces. Furthermore, μ^* is finite (respectively, locally finite) if μ is finite (respectively, locally finite).

Proof. Given the measure μ_j in γ we define the measure μ_j^* in I by $\mu_j^*(A) := \mu_j(\gamma(A))$, for all Borel set $A \subseteq I$. This measure is well defined since γ is injective (if γ is a closed curve and its domain is $I = [a, b]$ we can consider $\gamma : [a, b) \to \mathbf{C}$ in order to define μ_j^*). With this definition we have that, for any function $f \in L^1(\gamma, \mu_j)$, $\int_\gamma f(z)\, d\mu_j(z) = \int_I f(\gamma(t))\, d\mu_j^*(t)$. We define now $\mu^* = (\mu_0^*, \ldots, \mu_k^*)$. It is clear that $\mu^* \in \mathrm{ESD}$ if $k \geqslant 2$, and μ^* is finite (respectively, locally finite) if μ is finite (respectively, locally finite). We have $w_j^* = |\gamma'|(w_j \circ \gamma)$; if γ is a closed curve and $I = [a, b]$, without loss of generality we can also assume that $\gamma(a) = \gamma(b)$ is a $(k-1)$-regular point; then we have that the set of j-regular points for μ, is the image by γ of the j-regular points for μ^*. This fact gives that μ^* is p-admissible and $(I, \mu^*) \in \mathscr{C}$. It is natural to define the linear mapping $\Phi : W^{k,p}(\gamma, \mu) \to W^{k,p}(I, \mu^*)$ given by $\Phi(f) = f \circ \gamma$. We shall see that Φ is an isomorphism.

Observe that $\Phi(f)' = f'(\gamma)\gamma'$ and that

$$\Phi(f)^{(j)} = \sum_{i=1}^{j} f^{(i)}(\gamma) Q_{i,j}(\gamma) \qquad \text{for} \quad 1 \leqslant j \leqslant k,$$

where $Q_{i,j}$ is a differential operator of degree less than or equal to j. As $\gamma^{(i)} \in L^\infty(I)$ for $1 \leqslant i \leqslant k$, we obtain

$$\begin{aligned} \|\Phi(f)^{(j)}\|_{L^p(I,\mu_j^*)} &\leqslant c \sum_{i=1}^{j} \|f^{(i)}(\gamma)\|_{L^p(I,\mu_j^*)} \\ &= c \sum_{i=1}^{j} \|f^{(i)}\|_{L^p(\gamma,\mu_j)} \leqslant c \sum_{i=1}^{j} \|f^{(i)}\|_{L^p(\gamma,\mu_i)} \\ &\leqslant c\|f\|_{W^{k,p}(\gamma,\mu)}, \end{aligned}$$

since $\mu \in \mathrm{ESD}$ if $k \geqslant 2$. That is to say $\|\Phi(f)\|_{W^{k,p}(I,\mu^*)} \leqslant c\|f\|_{W^{k,p}(\gamma,\mu)}$.

Since $(\gamma, \mu) \in \mathscr{C}$ and $(I, \mu^*) \in \mathscr{C}$, the other inequality is a consequence of the open mapping theorem in Banach spaces. ∎

6. DENSITY

We do not have a density theorem as general as Theorem 5.1, but Theorem 6.1 covers many important cases. We begin with the following definitions.

DEFINITION 6.1. Consider $1 \leqslant p < \infty$, a compact curve γ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in γ. We say that μ is of type 1 if it is finite and p-admissible in γ and $w_k \in B_p(\gamma)$.

DEFINITION 6.2. Consider $1 \leqslant p < \infty$, a non-closed compact curve $\gamma = [z_1, z_2]$ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in γ. We say that μ is of type 2 if it is finite and strongly p-admissible in γ and there exist points along the curve $z_1 \leqslant \zeta_1 < \zeta_2 < \zeta_3 < \zeta_4 \leqslant z_2$ and integers $k_1, k_2 \geqslant 0$ such that

(1) $w_k \in B_p([\zeta_1, \zeta_4])$,

(2) if $z_1 < \zeta_1$, then w_j is comparable to a non-decreasing weight in $[z_1, \zeta_2]$, for $k_1 \leqslant j \leqslant k$,

(3) if $\zeta_4 < z_2$, then w_j is comparable to a non-increasing weight in $[\zeta_3, z_2]$, for $k_2 \leqslant j \leqslant k$,

(4) z_1 is right $(k_1 - 1)$-regular if $k_1 > 0$ and z_2 is left $(k_2 - 1)$-regular if $k_2 > 0$.

DEFINITION 6.3. Consider $1 \leqslant p < \infty$, a compact curve γ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in γ. We say that μ is of type 3 if it is finite and p-admissible in γ and there exist $z_0 \in \gamma$, an open neighbourhood V of z_0 in γ, an integer $0 \leqslant r < k$ and a positive constant c such that

(1) $d\mu_{j+1}(z) \leqslant c|z - z_0|^p \, d\mu_j(z)$ in V, for $r \leqslant j < k$,

(2) $w_k \in B_p(\gamma \setminus \{z_0\})$,

(3) if $r > 0$, z_0 is $(r - 1)$-regular.

Remark. Condition (1) means that μ_{j+1} is absolutely continuous with respect to μ_j on V and its Radon–Nikodym derivative is less than or equal to $c|z - z_0|^p$.

DEFINITION 6.4. Consider $1 \leqslant p < \infty$, a compact curve γ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in γ. We say that μ is of type 4 if it is finite and p-admissible in γ and there exist $z_0 \in \gamma$, an open neighbourhood V of z_0 in γ and a positive constant c such that

(1) if $p > 1$, $w_k(z) \leqslant c|z - z_0|^{p-1}$ for almost every $z \in V$; if $p = 1$, w_k can be modified in a set of zero length in such a way that $\lim_{z \to z_0} w_k(z) = 0$,

(2) $w_k \in B_p(\gamma \setminus \{z_0\})$,

(3) if $k > 1$, z_0 is $(k - 2)$-regular.

DEFINITION 6.5. Consider $1 \leqslant p < \infty$, a non-closed compact curve $\gamma = [z_1, z_2]$ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in γ. We say that μ is of type 5 if it is finite and p-admissible in γ and verifies

(1) $w_k \in B_p((z_1, z_2))$,

(2) if $k > 1$, z_1 is right $(k - 2)$-regular and z_2 is left $(k - 2)$-regular.

Remark. We want to remark that the types of measures in [RARP2] and here do not coincide.

LEMMA 6.1. *Let us consider $1 \leqslant p < \infty$ and a finite p-admissible vectorial measure μ of type i $(1 \leqslant i \leqslant 5)$. Then there exists a finite vectorial p-admissible measure μ' of type i such that $\mu' \in$ ESD and $\mu' \geqslant \mu$.*

Proof. It is easy to check that the measure $\mu' = (\mu'_0, \ldots, \mu'_k)$ with $\mu'_j := \mu_j + \cdots + \mu_k$ verifies the required conditions. ∎

LEMMA 6.2. *Let us consider $1 \leqslant p < \infty$, $c > 0$, $\gamma : I \to \mathbf{C}$ a curve with $c^{-1} \leqslant |\gamma'| \leqslant c$ and $\gamma' \in W^{k-1,\infty}(I)$, and a vectorial measure μ of type i $(1 \leqslant i \leqslant 5)$, with $\mu \in$ ESD. Then the vectorial measure μ^* which appears in the statement of Theorem 5.2 is of type i.*

Proof. It is an immediate consequence of the following facts: $w_j^* = |\gamma'|(w_j \circ \gamma)$, $\|(w_j^*)^{-1}\|_{L^{1/(p-1)}(J)} \asymp \|w_j^{-1}\|_{L^{1/(p-1)}(\gamma(J))}$ for all arc $J \subseteq I$, γ is a bijection between the j-regular sets in $W^{k,p}(I, \mu^*)$ and $W^{k,p}(\gamma, \mu)$, and $|\gamma(t) - \gamma(t_0)| \leqslant c|t - t_0|$. ∎

THEOREM 6.1. *Let us consider $1 \leqslant p < \infty$, $c > 0$ and $\mu = (\mu_0, \ldots, \mu_k)$ a p-admissible vectorial measure in a compact curve $\gamma : I \to \mathbf{C}$. Let us assume that $c^{-1} \leqslant |\gamma'| \leqslant c$ and $\gamma' \in W^{k-1,\infty}(I)$. If μ is a measure of type 1, 2, 3, 4 or 5, then $AC^k(I)$ is dense in the Sobolev space $W^{k,p}(\gamma, \mu)$. Furthermore, if $\gamma \in C^\infty(I)$, then $C^\infty(\gamma)$ is dense in $W^{k,p}(\gamma, \mu)$.*

Proof. Assume first that γ is not a closed curve. We can replace the measure μ by a greater measure, since then the theorem is more difficult. Therefore, by Lemmas 6.1 and 6.2, we can assume $\mu \in$ ESD, and so the measure μ^* which appears in the statement of Theorem 5.2 is of type i.

We can deduce that $C^\infty(\mathbf{R})$ is dense in $W^{k,p}(I, \mu^*)$; this is an immediate consequence of [RARP2, Theorem 4.1] if μ is a measure of type 1, 2 or 4. On the other hand, if μ is of type 3 (respectively, 5) this fact follows from [R3, Theorem 3.4] (respectively, [R3, Theorem 3.3]). Recall that the types of measures in [RARP2] and here do not coincide.

Therefore $AC^k(I)$ is dense in $W^{k,p}(I, \mu^*)$. By Theorem 5.2 and Lemma 2.7, $AC^k(\gamma)$ is dense in $W^{k,p}(\gamma, \mu)$. If $\gamma \in C^\infty(I)$, Theorem 5.2 gives that $C^\infty(\gamma)$ is dense in $W^{k,p}(\gamma, \mu)$. This finishes the proof in this case.

If γ is closed the proof is similar but it is necessary to reformulate slightly the last arguments. As an example we deal now with type 1.

Let $f \in V^{k,p}(\gamma, \mu)$. Let g be a continuous function in γ which approximates $f^{(k)}$ in the $L^p(\gamma, \mu_k)$ norm. Fix $z_0 \in \gamma$ and consider the function

$$h(z) := \sum_{j=0}^{k-1} f^{(j)}(z_0) \frac{(z - z_0)^j}{j!} + \int_{z_0}^{z} g(\zeta) \frac{(z - \zeta)^{k-1}}{(k-1)!} d\zeta.$$

We have, for $0 \leqslant j < k$, that

$$||f^{(j)} - h^{(j)}||_{L^p(\gamma,\mu_j)} \leqslant c||f^{(k)} - g||_{L^1(\gamma)} \leqslant c||f^{(k)} - g||_{L^p(\gamma,\mu_k)},$$

and then

$$||f - h||_{W^{k,p}(\gamma,\mu)} \leqslant c||f^{(k)} - g||_{L^p(\gamma,\mu_k)} \qquad \text{with } h \in AC^k(\gamma). \quad \blacksquare$$

THEOREM 6.2. *Let us consider $1 \leqslant p < \infty$, $c > 0$ and $\mu = (\mu_0, \ldots, \mu_k)$ a p-admissible vectorial measure in a non-closed compact curve $\gamma : I \to \mathbf{C}$. Let us assume that $c^{-1} \leqslant |\gamma'| \leqslant c$ and $\gamma' \in W^{k-1,\infty}(I)$. If μ is a measure of type* $1, 2, 3, 4$ *or* 5, *then P is dense in the Sobolev space $W^{k,p}(\gamma, \mu)$.*

Proof. Let $f_0 \in V^{k,p}(\gamma, \mu)$. By Theorem 6.1 we can approximate f_0 by a function $f \in AC^k(\gamma)$. Let g be a continuous function approximating $f^{(k)}$ in the $L^p(\gamma, \mu_k)$ and the $L^1(\gamma)$ norms (see [R3, Lemma 3.1]). Since γ is non-closed, we can choose a polynomial q approximating g in $L^\infty(\gamma)$ (and therefore in the $L^p(\gamma, \mu_k)$ and the $L^1(\gamma)$ norms). By fixing $z_0 \in \gamma$ and considering the function

$$Q(z) := \sum_{j=0}^{k-1} f^{(j)}(z_0) \frac{(z - z_0)^j}{j!} + \int_{z_0}^{z} q(\zeta) \frac{(z - \zeta)^{k-1}}{(k-1)!} d\zeta,$$

we can finish the proof as above. $\blacksquare$

7. DENSITY IN ANALYTIC CLOSED CURVES

We deal first with the case of the unit circle $\partial\mathbf{D}$.

LEMMA 7.1. *Let us consider $1 \leqslant p < \infty$, $m \in \mathbf{Z}^+$ and μ a finite scalar measure in $\partial\mathbf{D}$. Then the polynomials P are dense in $L^p(\partial\mathbf{D}, \mu)$ if and only if $1/z^m$ belongs to the closure of P in $L^p(\partial\mathbf{D}, \mu)$.*

Proof. We prove first the result for $m = 1$. The "only if" direction is immediate. In order to prove the non-trivial implication, assume that $1/z$

belongs to the closure of P. Then we have, for any $r, n \in \mathbf{Z}^+$,

$$\begin{aligned}\operatorname{dist}(1/z, P_n)^p &:= \min_{a_i \in \mathbf{C}} \int_{\partial \mathbf{D}} |z^{-1} - (a_0 + a_1 z + \cdots + a_n z^n)|^p \, d\mu(z) \\ &= \min_{a_i \in \mathbf{C}} \int_{\partial \mathbf{D}} |z^{-r} - (a_0 z^{1-r} + a_1 z^{2-r} + \cdots + a_n z^{n+1-r})|^p \, d\mu(z) \\ &= \operatorname{dist}(z^{-r}, \operatorname{span}\{z^{1-r}, z^{2-r}, \ldots, z^{n+1-r}\})^p.\end{aligned}$$

This fact and an induction argument in r give that $1/z^r$ belongs to the closure of P in $L^p(\partial \mathbf{D}, \mu)$, for every $r \in \mathbf{Z}^+$. Since any function in $L^p(\partial \mathbf{D}, \mu)$ can be approximated by continuous functions in the norm of $L^p(\partial \mathbf{D}, \mu)$, and that any continuous function can be approximated uniformly in $\partial \mathbf{D}$ by linear combinations of integer powers of z, we have that the polynomials are dense in $L^p(\partial \mathbf{D}, \mu)$.

We prove now that $1/z^m$ belongs to the closure of P if and only if $1/z$ belongs to the closure of P. The last argument gives that $1/z^m$ belongs to the closure of P if $1/z$ does. Assume now that $1/z^m$ belongs to the closure of P. Choose $p_n \in P$ with $||p_n - 1/z^m||_{L^p(\mu)} \to 0$. Then

$$||z^{m-1} p_n - 1/z||_{L^p(\mu)} = ||p_n - 1/z^m||_{L^p(\mu)} \to 0,$$

so $1/z$ belongs to the closure of P, and this finishes the proof of the lemma. ∎

Proposition 7.1. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a finite p-admissible vectorial measure in* $\partial \mathbf{D}$. *If the polynomials are dense in* $W^{k,p}(\partial \mathbf{D}, \mu)$, *then they are dense in* $L^p(\partial \mathbf{D}, \mu_j)$ *for any* $0 \leqslant j \leqslant k$.

Proof. Fix $0 \leqslant j \leqslant k$. The function $1/z$ can be approximated by polynomials in $W^{k,p}(\partial \mathbf{D}, \mu)$. Then the function $1/z^{j+1}$ can be approximated by polynomials in $L^p(\partial \mathbf{D}, \mu_j)$. Lemma 7.1 gives now the result. ∎

Definition 7.1. A scalar measure μ in an analytic closed curve γ with absolutely continuous part w verifies the Szegö condition if

$$\int_\gamma \log w(z) |dz| > -\infty.$$

The following theorem of Kolmogorov–Krein–Szegö is well known (see, e.g. [G, pp. 135–137]).

Theorem A. *Let us consider* $1 \leqslant p < \infty$ *and a finite scalar measure* μ in $\partial \mathbf{D}$. *Then the polynomials are dense in* $L^p(\partial \mathbf{D}, \mu)$ *if and only if* μ *does not verify the Szegö condition.*

We have the following consequence of Proposition 7.1 and Theorem A.

COROLLARY 7.1. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a finite p-admissible vectorial measure* in $\partial\mathbf{D}$. *If for some* $0 \leqslant j \leqslant k$ *the measure* μ_j *verifies the Szegö condition, then the polynomials are not dense in* $W^{k,p}(\partial\mathbf{D}, \mu)$.

Remark. One could think that the converse of Corollary 7.1 is true. However, if we consider $A := \{z \in \partial\mathbf{D} : |\arg z| \leqslant \pi/2\}$, $B := \{z \in \partial\mathbf{D} : |\arg z| \geqslant \pi/4\}$ (with $\arg z \in (-\pi, \pi]$), $d\mu_0(z) := \chi_A(z)|dz|$ and $d\mu_1(z) := \chi_B(z)|dz|$, then μ_0, μ_1 do not verify the Szegö condition and the polynomials are not dense in $W^{1,p}(\partial\mathbf{D}, \mu)$, as the following results, which are improvements of Corollary 7.1, show.

THEOREM 7.1. *Let us consider* $1 \leqslant p < \infty$, $\mu = (\mu_0, \dots, \mu_k)$ *a finite p-admissible vectorial measure in* $\partial\mathbf{D}$ *with* $(\partial\mathbf{D}, \mu) \in \mathscr{C}_0$ *and* $\overline{\mu}$ *a finite sum of completions of* μ. *If for some* $0 \leqslant j \leqslant k$ *the measure* $\overline{\mu}_j$ *verifies the Szegö condition, then the polynomials are not dense in* $W^{k,p}(\partial\mathbf{D}, \mu)$.

Proof. Part (a) of Theorem 4.1 and the fact $\overline{\mu} \geqslant \mu$ give that the polynomials are dense in $W^{k,p}(\partial\mathbf{D}, \mu)$ if and only if they are dense in $W^{k,p}(\partial\mathrm{D}, \overline{\mu})$. Now Corollary 7.1 gives the result. ∎

COROLLARY 7.2. *Let us consider* $1 \leqslant p < \infty$, *a fixed integer* $0 \leqslant j \leqslant k$, $\mu = (\mu_0, \dots, \mu_k)$ *a finite p-admissible vectorial measure in* $\partial\mathbf{D}$ *with* $(\partial\mathbf{D}, \mu) \in \mathscr{C}_0$ *and K a finite union of compact arcs with* $K \subseteq \Omega^{(j)}$. *If the measure* μ_j *verifies*

$$\int_{\partial\mathbf{D}\setminus K} \log w_j(z)|dz| > -\infty,$$

then the polynomials are not dense in $W^{k,p}(\partial\mathbf{D}, \mu)$.

Proof. Theorem 4.1 guarantees that we can take a measure $\overline{\mu}$, as in Theorem 7.1, such that $\overline{w}_j(z) \geqslant w_j(z) + \chi_K(z)$. Then we only need to apply Theorem 7.1. ∎

As positive results on density of polynomials in $\partial\mathbf{D}$ we have already proved the theorems in Section 6 when Δ, the union of the supports of μ_j, is not equal to $\partial\mathbf{D}$ (it is enough to consider a non-closed curve γ with $\Delta \subseteq \gamma$).

We deal now with general analytic closed curves.

PROPOSITION 7.2. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, \ldots, \mu_k)$ a finite p-admissible vectorial measure in an analytic closed curve γ. Let us assume that $\mu \in$ ESD if $k \geqslant 2$. If for some $0 \leqslant j \leqslant k$ the measure μ_j verifies the Szegö condition, then the polynomials are not dense in $W^{k,p}(\gamma, \mu)$.*

Proof. Let us consider a conformal map F between $\mathbf{D}$ and the simply connected domain E bounded by γ. Since γ is analytic, we can extend F to $\partial\mathbf{D}$ with $F : \bar{\mathbf{D}} \to \bar{E}$ biholomorphic.

Given the measure μ_j in γ we define the measure μ_j^* in $\partial\mathbf{D}$ by $\mu_j^*(A) := \mu_j(F(A))$, for all Borel set $A \subseteq \partial\mathbf{D}$. Since $\mu \in$ ESD if $k \geqslant 2$, the argument in the proof of Theorem 5.2 gives that $W^{k,p}(\partial\mathbf{D}, \mu^*)$ and $W^{k,p}(\gamma, \mu)$ are isomorphic as normed spaces. By Mergelyan and Weierstrass theorems the polynomials are dense in $W^{k,p}(\gamma, \mu)$ if and only if the holomorphic functions in $\bar{E}$ are dense in $W^{k,p}(\gamma, \mu)$. Therefore P is dense in $W^{k,p}(\partial\mathbf{D}, \mu^*)$ if and only it is dense in $W^{k,p}(\gamma, \mu)$. Since $w_j^* = |F'|(w_j \circ F)$ and there is a positive constant c with $c^{-1} \leqslant |F'| \leqslant c$ in $\partial\mathbf{D}$, μ_j verifies the Szegö condition if and only if μ_j^* does. These facts and Corollary 7.1 give the result. ∎

The same argument used in the proof of Proposition 7.2 gives the following generalization of the theorem of Kolmogorov–Helson–Szegö.

COROLLARY 7.3. *Let us consider $1 \leqslant p < \infty$ and a finite scalar measure μ in an analytic closed curve γ. Then the polynomials are dense in $L^p(\gamma, \mu)$ if and only if μ does not verify the Szegö condition.*

The same proof of Theorem 7.1 and Corollary 7.2 (using now Proposition 7.2) gives the following results.

THEOREM 7.2. *Let us consider $1 \leqslant p < \infty, \mu = (\mu_0, \ldots, \mu_k)$ a finite p-admissible vectorial measure in an analytic closed curve γ, with $(\gamma, \mu) \in \mathscr{C}_0$ and $\bar{\mu}$ a finite sum of completions of μ. Let us assume that $\mu \in$ ESD if $k \geqslant 2$. If for some $0 \leqslant j \leqslant k$ the measure $\overline{\mu}_j$ verifies the Szegö condition, then the polynomials are not dense in $W^{k,p}(\gamma, \mu)$.*

THEOREM 7.3. *Let us consider $1 \leqslant p < \infty$, a fixed integer $0 \leqslant j \leqslant k$, $\mu = (\mu_0, \ldots, \mu_k)$ a finite p-admissible vectorial measure in an analytic closed curve γ, with $(\gamma, \mu) \in \mathscr{C}_0$ and K a finite union of compact arcs with $K \subseteq \Omega^{(j)}$. Let us assume that $\mu \in$ ESD if $k \geqslant 2$. If the measure μ_j verifies*

$$\int_{\gamma \setminus K} \log w_j(z)|dz| > -\infty,$$

then the polynomials are not dense in $W^{k,p}(\gamma, \mu)$.

8. MULTIPLICATION OPERATOR

First of all, some remarks about the definition of the multiplication operator. In this section we only consider measures such that every polynomial has finite Sobolev norm. Recall that when every polynomial has finite $W^{k,p}(E,\mu)$-norm, we denote by $P^{k,p}(E,\mu)$ the completion of P with that norm. We start with a definition which has sense for measures defined in arbitrary Borel sets (not necessarily curves).

Definition 8.1. If μ is a vectorial measure in the Borel set $E \subseteq \mathbf{C}$, we say that the multiplication operator is well defined in $P^{k,p}(E,\mu)$ if given any sequence $\{s_n\}$ of polynomials converging to 0 in $W^{k,p}(E,\mu)$, then $\{zs_n\}$ also converges to 0 in $W^{k,p}(E,\mu)$. In this case, if $\{q_n\} \in P^{k,p}(E,\mu)$, we define $M(\{q_n\}) := \{zq_n\}$. If we choose another Cauchy sequence $\{r_n\}$ representing the same element in $P^{k,p}(E,\mu)$ (i.e. $\{q_n - r_n\}$ converges to 0 in $W^{k,p}(E,\mu)$), then $\{zq_n\}$ and $\{zr_n\}$ represent the same element in $P^{k,p}(E,\mu)$ (since $\{z(q_n - r_n)\}$ converges to 0 in $W^{k,p}(E,\mu)$).

We can also think of another definition which is as natural in the case of curves:

Definition 8.2. If μ is a p-admissible vectorial measure in γ (and hence $W^{k,p}(\gamma,\mu)$ is a space of classes of functions), we say that the multiplication operator is well defined in $W^{k,p}(\gamma,\mu)$ if given any function $h \in V^{k,p}(\gamma,\mu)$ with $||h||_{W^{k,p}(\gamma,\mu)} = 0$, we have $||zh||_{W^{k,p}(\gamma,\mu)} = 0$. In this case, if $[f]$ is an equivalence class in $W^{k,p}(\gamma,\mu)$, we define $M([f]) := [zf]$. If we choose another representative g of $[f]$ (i.e. $||f-g||_{W^{k,p}(\gamma,\mu)} = 0$) we have $[zf] = [zg]$, since $||z(f-g)||_{W^{k,p}(\gamma,\mu)} = 0$.

Although both definitions are natural, it is possible for a p-admissible measure μ with $W^{k,p}(\gamma,\mu) = P^{k,p}(\gamma,\mu)$ that M is well defined in $W^{k,p}(\gamma,\mu)$ and not well defined in $P^{k,p}(\gamma,\mu)$ (see Lemma 8.1 and Theorem 8.3). The following lemma characterizes the spaces $P^{k,p}(E,\mu)$ with M well defined.

Lemma 8.1. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, \dots, \mu_k)$ a vectorial measure in a Borel set $E \subseteq \mathbf{C}$. The following facts are equivalent*:

(1) *The multiplication operator is well defined in $P^{k,p}(E,\mu)$.*

(2) *The multiplication operator is bounded in $P^{k,p}(E,\mu)$.*

(3) *There exists a positive constant c such that*
$$||zq||_{W^{k,p}(E,\mu)} \leqslant c||q||_{W^{k,p}(E,\mu)} \qquad \textit{for every } q \in P.$$

Remark. When we say that the multiplication operator is bounded in $P^{k,p}(E,\mu)$, we assume implicitly that it is well defined in $P^{k,p}(E,\mu)$, since otherwise the boundedness has no sense.

Proof. It is clear that condition (3) implies (1). If we assume (1), we have that the multiplication operator M is continuous in $0 \in (P, ||\cdot||_{W^{k,p}(E,\mu)})$. Since M is a linear operator in the normed space $(P, ||\cdot||_{W^{k,p}(E,\mu)})$, we know that M is bounded in $(P, ||\cdot||_{W^{k,p}(E,\mu)})$, which gives (3).

We now show the equivalence between (2) and (3). Let us consider an element $\alpha \in P^{k,p}(E,\mu)$. This element α is an equivalence class of Cauchy sequences of polynomials under the norm in $W^{k,p}(E,\mu)$. Assume that a Cauchy sequence of polynomials $\{q_n\}$ represents α. The norm of α is defined as $||\alpha||_{P^{k,p}(E,\mu)} = \lim_{n\to\infty} ||q_n||_{W^{k,p}(E,\mu)}$, which obviously does not depend on the representative. Hence, condition (2) is equivalent to

$$\lim_{n\to\infty} ||zq_n||_{W^{k,p}(E,\mu)} \leqslant c \lim_{n\to\infty} ||q_n||_{W^{k,p}(E,\mu)},$$

for every Cauchy sequence of polynomials $\{q_n\}$. Now the equivalence between (2) and (3) is clear. ∎

LEMMA 8.2. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, \ldots, \mu_k)$ a finite vectorial measure in a compact set E. Then, the multiplication operator is bounded in $P^{k,p}(E,\mu)$ if and only if there exists a positive constant c such that*

$$||q^{(j-1)}||_{L^p(E,\mu_j)} \leqslant c||q||_{W^{k,p}(E,\mu)}$$

for every $1 \leqslant j \leqslant k$ and $q \in P$.

Proof. If M is bounded in $P^{k,p}(E,\mu)$, we have that

$$||(zq)^{(j)}||_{L^p(E,\mu_j)} \leqslant ||M|| ||q||_{W^{k,p}(E,\mu)}$$

for every $1 \leqslant j \leqslant k$ and $q \in P$. Since

$$||(zq)^{(j)}||_{L^p(E,\mu_j)} = ||zq^{(j)} + jq^{(j-1)}||_{L^p(E,\mu_j)} \geqslant ||q^{(j-1)}||_{L^p(E,\mu_j)} - K||q^{(j)}||_{L^p(E,\mu_j)},$$

with $K := \max\{|z| : z \in E\}$, we have

$$||q^{(j-1)}||_{L^p(E,\mu_j)} \leqslant K||q^{(j)}||_{L^p(E,\mu_j)} + ||M||\ ||q||_{W^{k,p}(E,\mu)} \leqslant (K + ||M||)||q||_{W^{k,p}(E,\mu)}$$

for every $1 \leqslant j \leqslant k$ and $q \in P$.

We now prove the converse implication. Observe that

$$||(zq)^{(j)}||_{L^p(E,\mu_j)} = ||zq^{(j)} + jq^{(j-1)}||_{L^p(E,\mu_j)} \leqslant j||q^{(j-1)}||_{L^p(E,\mu_j)} + K||q^{(j)}||_{L^p(E,\mu_j)},$$

with K as before, for every $1 \leqslant j \leqslant k$ and $q \in P$. Then

$$\begin{aligned} ||(zq)^{(j)}||^p_{L^p(E,\mu_j)} &\leqslant 2^{p-1}(j^p||q^{(j-1)}||^p_{L^p(E,\mu_j)} + K^p||q^{(j)}||^p_{L^p(E,\mu_j)}) \\ &\leqslant 2^{p-1}(j^p c^p||q||^p_{W^{k,p}(E,\mu)} + K^p||q^{(j)}||^p_{L^p(E,\mu_j)}) \end{aligned}$$

for every $1 \leqslant j \leqslant k$ and $q \in P$ (if $j = 0$ the inequality is trivial). Consequently

$$||zq||^p_{W^{k,p}(E,\mu)} \leqslant 2^{p-1}(k^{p+1}c^p||q||^p_{W^{k,p}(E,\mu)} + K^p||q||^p_{W^{k,p}(E,\mu)})$$

and

$$||zq||_{W^{k,p}(E,\mu)} \leqslant 2^{(p-1)/p}(k^{p+1}c^p + K^p)^{1/p}||q||_{W^{k,p}(E,\mu)}$$

for every $q \in P$. Hence, Lemma 8.1 proves that M is bounded in $P^{k,p}(E,\mu)$. ∎

In the following we often use the next result. We omit the proof since it is elementary.

LEMMA 8.3. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$, $\mu' = (\mu'_0, \ldots, \mu'_k)$ *vectorial measures in a Borel set* $E \subseteq \mathbf{C}$. *If the Sobolev norms in* $W^{k,p}(E,\mu)$ *and* $W^{k,p}(E,\mu')$ *are comparable on* P, *then*:

(1) $P^{k,p}(E,\mu) = P^{k,p}(E,\mu')$.

(2) M *is bounded in* $P^{k,p}(E,\mu)$ *if and only if it is bounded in* $P^{k,p}(E,\mu')$.

THEOREM 8.1. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a finite vectorial measure in a compact set* E. *Then, the multiplication operator is bounded in* $P^{k,p}(E,\mu)$ *if and only if there exists a vectorial measure* $\mu' \in$ ESD *such that the Sobolev norms in* $W^{k,p}(E,\mu)$ *and* $W^{k,p}(E,\mu')$ *are comparable on* P. *Furthermore, we can choose* $\mu' = (\mu_0, \ldots, \mu'_k)$ *with* $\mu'_j := \mu_j + \mu_{j+1} + \cdots + \mu_k$.

Remark. In order to apply Theorem 8.1, if $E = \gamma$ is a curve, the best way to deduce that $||\cdot||_{W^{k,p}(\gamma,\mu)}$ and $||\cdot||_{W^{k,p}(\gamma,\mu')}$ are equivalent is to prove that μ' can be obtained by a finite number of completions of μ (in that case we can use Theorem 4.1).

Proof. Assume that there exists a vectorial measure $\mu' \in \mathrm{ESD}$ such that the Sobolev norms in $W^{k,p}(E,\mu)$ and $W^{k,p}(E,\mu')$ are comparable on P. By Lemmas 8.2 and 8.3 it is enough to show

$$||q^{(j-1)}||_{L^p(E,\mu'_j)} \leqslant c||q||_{W^{k,p}(E,\mu')} \tag{8.1}$$

for every $1 \leqslant j \leqslant k$ and $q \in P$. The hypothesis $\mu' \in \mathrm{ESD}$ gives

$$||q^{(j-1)}||_{L^p(E,\mu'_j)} \leqslant c||q^{(j-1)}||_{L^p(E,\mu'_{j-1})} \leqslant c||q||_{W^{k,p}(E,\mu')},$$

and then we have (8.1).

Assume now that M is bounded in $P^{k,p}(E,\mu)$ and let us consider the vectorial measures $\mu^0, \mu^1, \ldots, \mu^{k-1}$, μ^k defined by

$$\mu^j_i := \mu_i \qquad \text{if } 0 \leqslant i < j,$$

$$\mu^j_i := \sum_{l=i}^{k} \mu_l \qquad \text{if } j \leqslant i \leqslant k.$$

Observe that $\mu^k = \mu$ and μ^0 is the measure μ' defined at the end of the statement of Theorem 8.1. These vectorial measures verify, for $0 \leqslant i \leqslant k$ and $0 < j \leqslant k$,

$$\mu^{j-1}_i := \mu^j_i \qquad \text{if } i \neq j-1, \tag{8.2}$$

$$\mu^{j-1}_{j-1} := \mu^j_j + \mu_{j-1} = \mu^j_j + \mu^j_{j-1}. \tag{8.3}$$

Therefore we have $||q||_{W^{k,p}(E,\mu^j)} \leqslant ||q||_{W^{k,p}(E,\mu^{j-1})}$, for every $q \in P$ and $1 \leqslant j \leqslant k$.

Since $\mu^0 \in \mathrm{ESD}$ it is enough to show that the Sobolev norms in $W^{k,p}(E, \mu^k)$ and $W^{k,p}(E,\mu^0)$ are comparable on P. We prove this by showing for $1 \leqslant j \leqslant k$ that the Sobolev norms in $W^{k,p}(E,\mu^j)$ and $W^{k,p}(E,\mu^{j-1})$ are comparable on P and M is bounded in $P^{k,p}(E,\mu^{j-1})$. We prove this last statement by reverse induction on j. Assume that the induction hypothesis holds for $j+1$. Then we have that M is bounded in $P^{k,p}(E,\mu^j)$. Lemma 8.2 gives that

$$||q^{(j-1)}||_{L^p(E,\mu^j_j)} \leqslant c||q||_{W^{k,p}(E,\mu^j)}$$

for every $q \in P$. This inequality and (8.3) show

$$||q^{(j-1)}||^p_{L^p(E,\mu^{j-1}_{j-1})} \leqslant c^p||q||^p_{W^{k,p}(E,\mu^j)} + ||q^{(j-1)}||^p_{L^p(E,\mu^j_{j-1})} \leqslant (c^p+1)||q||^p_{W^{k,p}(E,\mu^j)}$$

for every $q \in P$. This fact and (8.2) show that the Sobolev norms in $W^{k,p}(E, \mu^j)$ and $W^{k,p}(E, \mu^{j-1})$ are comparable on P. Then Lemma 8.3 shows that M is bounded in $P^{k,p}(E, \mu^{j-1})$, since it is bounded in $P^{k,p}(E, \mu^j)$. The proof of the case $j = k$ is similar. This finishes the induction argument and the proof of Theorem 8.1. ∎

If we consider the case of a curve $E = \gamma$, we have the following results.

THEOREM 8.2. *Let us consider* $1 \leqslant p < \infty$ *and a p-admissible vectorial measure* μ *in* γ. *If* μ *is of type* 1, 2 *or* 3, *and the multiplication operator is well defined in* $W^{k,p}(\gamma, \mu)$, *then it is bounded in* $P^{k,p}(\gamma, \mu)$.

Remark. In this situation Theorem 6.2 gives $P^{k,p}(\gamma, \mu) = W^{k,p}(\gamma, \mu)$ if $\gamma : I \to \mathbf{C}$ is a non-closed curve with $c^{-1} \leqslant |\gamma'| \leqslant c$ and $\gamma' \in W^{k-1,\infty}(I)$. In this case the multiplication operator is bounded in $W^{k,p}(\gamma, \mu)$.

Obviously, the multiplication operator M is well defined in $W^{k,p}(\gamma, \mu)$ if and only if it is well defined in $V^{k,p}(\gamma, \mu)$ (i.e. $zf \in V^{k,p}(\gamma, \mu)$ for every $f \in V^{k,p}(\gamma, \mu)$) and $||zf||_{W^{k,p}(\gamma,\mu)} = 0$ for every $f \in V^{k,p}(\gamma, \mu)$ with $||f||_{W^{k,p}(\gamma,\mu)} = 0$. This second condition can be written as $M(\mathscr{K}(\gamma, \mu)) \subseteq \mathscr{K}(\gamma, \mu)$.

THEOREM 8.3. *Let us consider* $1 \leqslant p < \infty$ *and a p-admissible vectorial measure* μ *in* γ. *Assume that the multiplication operator* M *is well defined in* $V^{k,p}(\gamma, \mu)$. *Then* M *is well-defined in* $W^{k,p}(\gamma, \mu)$ *if and only if* $\mathscr{K}(\gamma, \mu) = \{0\}$.

Proof. Suppose first that $\mathscr{K}(\gamma, \mu) = \{0\}$. Then, if $f \in V^{k,p}(\gamma, \mu)$ with $||f||_{W^{k,p}(\gamma,\mu)} = 0$ we have $||f||_{W^{k,p}(\gamma,\mu|_{\Omega}(0))}) = 0$. Consequently $f|_{\Omega^{(0)}} \equiv 0$ and so $||zf||_{W^{k,p}(\gamma,\mu|_{\Omega(0)})} = 0$. On the other hand, we also have $||f||_{L^p(\gamma,\mu_0)} = 0$, and so $f(z) = 0$ for μ_0-almost every $z \in \gamma$. Then $zf(z) = 0$ for μ_0-almost every $z \in \gamma$ and $||zf||_{L^p(\gamma,\mu_0)} = 0$. Observe that μ_j is concentrated in $\Omega_j \cup \Omega^{(j)} \subseteq \Omega^{(0)}$ for $1 \leqslant j \leqslant k$. We deduce from these facts that

$$||zf||^p_{W^{k,p}(\gamma,\mu)} \leqslant ||zf||^p_{L^p(\gamma,\mu_0)} + ||zf||^p_{W^{k,p}(\gamma,\mu|_{\Omega(0)})} = 0,$$

and therefore the multiplication operator is well defined in $W^{k,p}(\gamma, \mu)$.

On the converse, let us suppose that there is $f \in V^{k,p}(\gamma, \mu)$ such that

$$||f||_{W^{k,p}(\gamma,\mu|_{\Omega(0)})} = 0,$$

but f is not identically zero in $\Omega^{(0)}$. We know that there exists an arc $\gamma_0 \subseteq \Omega^{(0)}$ such that $f|_{\gamma_0} \neq 0$, and therefore there is another arc $\gamma_1 \subseteq \gamma_0$ such that $\gamma_1 \subseteq \Omega_i$ for some $1 \leqslant i \leqslant k$ and $f|_{\gamma_1} \neq 0$. If g belongs to $\mathscr{K}(\gamma, \mu)$, we have that $g^{(i)}(z) = 0$ for almost every $z \in \Omega_i$, and therefore that $g^{(i-1)}$ is constant in each connected component of Ω_i. Then $g|_{\gamma_1} \in P_{i-1}$. Let us choose now $h \in \mathcal{K}(\gamma, \mu)$ such that $\deg h|_{\gamma_1} \geqslant \deg g|_{\gamma_1}$ for all $g \in \mathcal{K}(\gamma, \mu)$ (we have $\deg h|_{\gamma_1} \geqslant 0$ since the function f is not identically zero in γ_1). Then, $\deg zh|_{\gamma_1} > \deg h|_{\gamma_1}$; therefore $zh \notin \mathcal{K}(\gamma, \mu)$ and M is not well defined. ∎

Proof of Theorem 8.2. We divide this proof into three parts; each of them will be devoted to each type of measure. Remember that in our hypotheses we always have $\mathcal{K}(\gamma, \mu) = \{0\}$ by Theorem 8.3. Therefore $(\gamma, \mu) \in \mathcal{C}_0$, since $\Omega^{(0)} \setminus (\Omega_1 \cup \cdots \cup \Omega_k)$ has at most two points (see Remark 1 after Definition 4.2).

Measures of type 1. By Theorem 4.1 we have directly

$$||f^{(j-1)}||_{L^p(\gamma,\mu_j)} \leqslant c||f^{(j-1)}||_{L^\infty(\gamma)} \leqslant c||f||_{W^{k,p}(\gamma,\mu)}$$

for all $f \in V^{k,p}(\gamma, \mu)$ and $1 \leqslant j \leqslant k$, since $(\gamma, \mu) \in \mathcal{C}_0$. Now Lemma 8.2 gives the conclusion.

Measures of type 2. A computation (using Muckenhoupt inequality) gives that

$$c||f^{(j-1)}||_{L^p([z_1,\zeta_2],\mu_j)} \leqslant ||f^{(j)}||_{L^p([z_1,\zeta_2],\mu_j)} + |f^{(j-1)}(\zeta_2)|$$

for $k_1 \leqslant j \leqslant k$. Then Theorem 4.1 gives

$$||f^{(j-1)}||_{L^p([z_1,\zeta_2],\mu_j)} \leqslant c||f||_{W^{k,p}(\gamma,\mu)}. \tag{8.4}$$

If $k_1 > 1$, again by Theorem 4.1, we have

$$||f^{(j-1)}||_{L^p([z_1,\zeta_2],\mu_j)} \leqslant c||f^{(j-1)}||_{L^\infty([z_1,\zeta_2])} \leqslant c||f||_{W^{k,p}(\gamma,\mu)},$$

for all $f \in V^{k,p}(\gamma, \mu)$ and $1 \leqslant j < k_1$, since z_1 is right $(k_1 - 1)$-regular (and then $[z_1, \zeta_2] \subseteq \Omega^{(k_1-1)}$). Therefore (8.4) is true for all $f \in V^{k,p}(\gamma, \mu)$ and $1 \leqslant j \leqslant k$. The arc $[\zeta_3, z_2]$ is treated in a symmetric way and we obtain an inequality similar to (8.4). The arc $[\zeta_2, \zeta_3]$ needs the same argument as measures of type 1.

Measures of type 3. Condition (1) of measures of type 3 gives

$$||f^{(j-1)}||_{L^p(V,\mu_j)} \leqslant c||f^{(j-1)}||_{L^p(V,\mu_{j-1})}$$

for $r<j\leqslant k$. If $r>0$, Theorem 4.1 gives

$$\|f^{(j-1)}\|_{L^p(V,\mu_j)}\leqslant c\|f^{(j-1)}\|_{L^\infty(V)}\leqslant c\|f\|_{W^{k,p}(\gamma,\mu)}$$

for $1\leqslant j\leqslant r$. Consequently, we have

$$\|f^{(j-1)}\|_{L^p(V,\mu_j)}\leqslant c\|f\|_{W^{k,p}(\gamma,\mu)},$$

for $1\leqslant j\leqslant k$. The arc $\gamma\setminus V$ needs the same argument as measures of type 1. ∎

THEOREM 8.4. *Let us consider $1\leqslant p<\infty$ and a finite p-admissible vectorial measure μ in a compact curve γ. Assume that $(\gamma,\mu)\in\mathscr{C}_0$ and that for each $1\leqslant j\leqslant k$ we have $\mu_j(\gamma\setminus(J_{j-1}\cup K_{j-1}))=0$, where K_{j-1} is a finite union of compact arcs contained in $\Omega^{(j-1)}$ and J_{j-1} is a Borel set with $\mu_j\leqslant c\mu_{j-1}$ in J_{j-1}. Then the multiplication operator is bounded in $P^{k,p}(\gamma,\mu)$.*

Proof. We have by $\mu_j(\gamma)<\infty$ and Theorem 4.1

$$\|g^{(j-1)}\|_{L^p(K_{j-1},\mu_j)}\leqslant c\|g^{(j-1)}\|_{L^\infty(K_{j-1})}\leqslant c\|g\|_{W^{k,p}(\gamma,\mu)}$$

for every $1\leqslant j\leqslant k$ and $g\in W^{k,p}(\gamma,\mu)$. The hypothesis on J_{j-1} gives

$$\|g^{(j-1)}\|_{L^p(J_{j-1},\mu_j)}\leqslant c\|g^{(j-1)}\|_{L^p(J_{j-1},\mu_{j-1})}\leqslant c\|g\|_{W^{k,p}(\gamma,\mu)}.$$

These two inequalities imply

$$\|g^{(j-1)}\|_{L^p(\gamma,\mu_j)}\leqslant c\|g\|_{W^{k,p}(\gamma,\mu)}$$

for every $1\leqslant j\leqslant k$ and $g\in W^{k,p}(\gamma,\mu)$. Lemma 8.2 finishes the proof. ∎

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